RELIABILITY ANALYSIS OF GEOTECHNICAL STRUCTURES INCLUDING TUNNELS, EMBANKMENTS AND RETAINING STRUCTURES

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Outline of the presentation

- Why probabilistic analysis?
- Simple Probabilistic Slope Stability Analysis in *Slide2*
- Probabilistic in Tunneling
- Spatial variability analysis
- Spatial variability analysis in *Slide2* 2018
- Spatial variability in Tunneling
- Reliability analysis of MSE walls

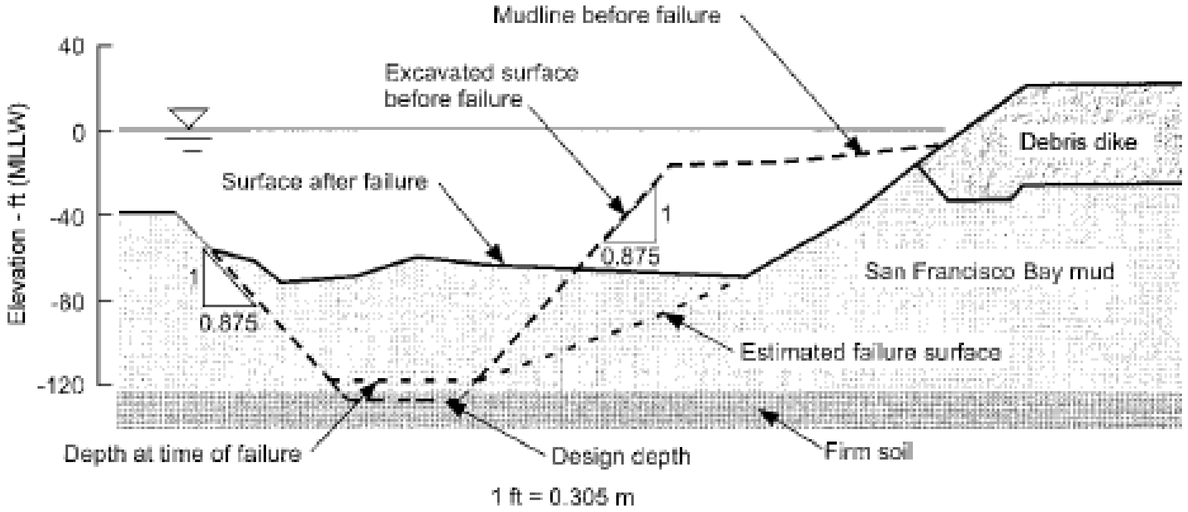
Why probabilistic analysis

Section 1

Problem Definition

LASH terminal at the Port of San Francisco (1970)

Temporary slope, cohesive soil, good laboratory data, F_s = 1.17



The slope was primarily designed for F_s = 1.17 → **Failed!!**

Post-failure probabilistic stability analysis → **Original P_f = 18%!!**

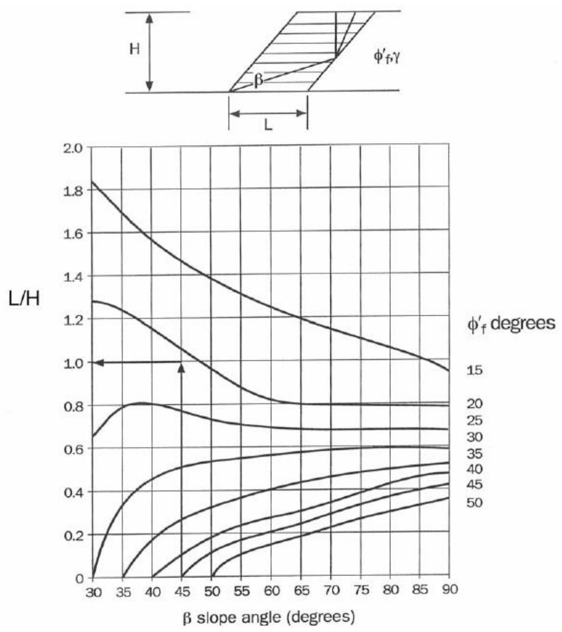
Problem definition

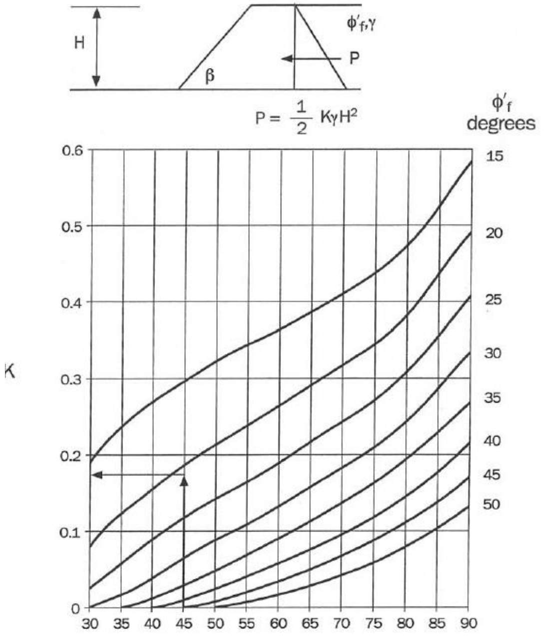
Deterministic Factor of safety (F_s): Limit equilibrium method, finite element method

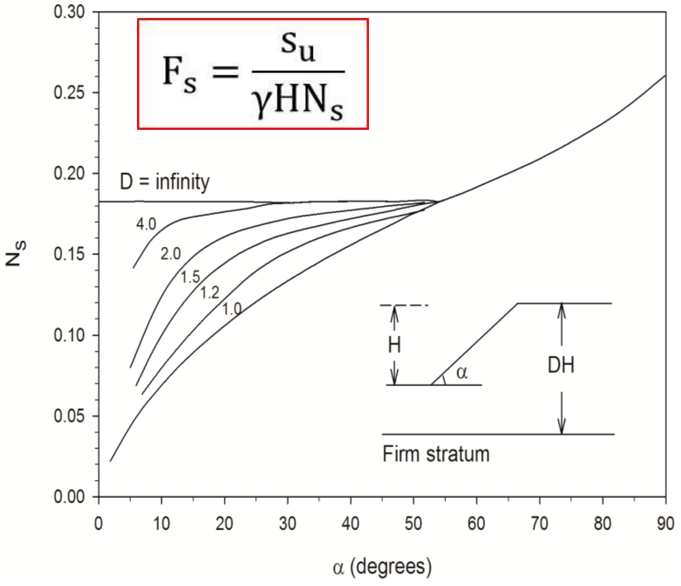
Stability charts are routinely used to calculate deterministic factor of safety in simple slopes

Taylor (1937): published stability charts for cohesive and cohesive-frictional soil slopes

Bathurst and Jones (2001): published design charts for reinforced slopes using two-part wedge method

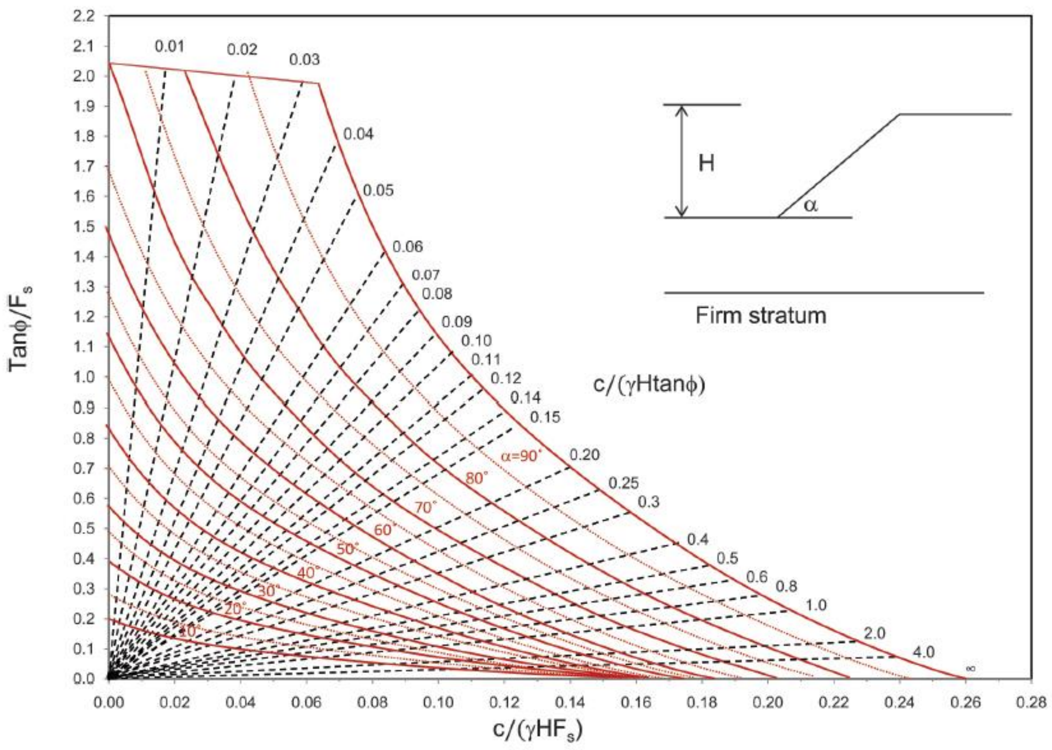






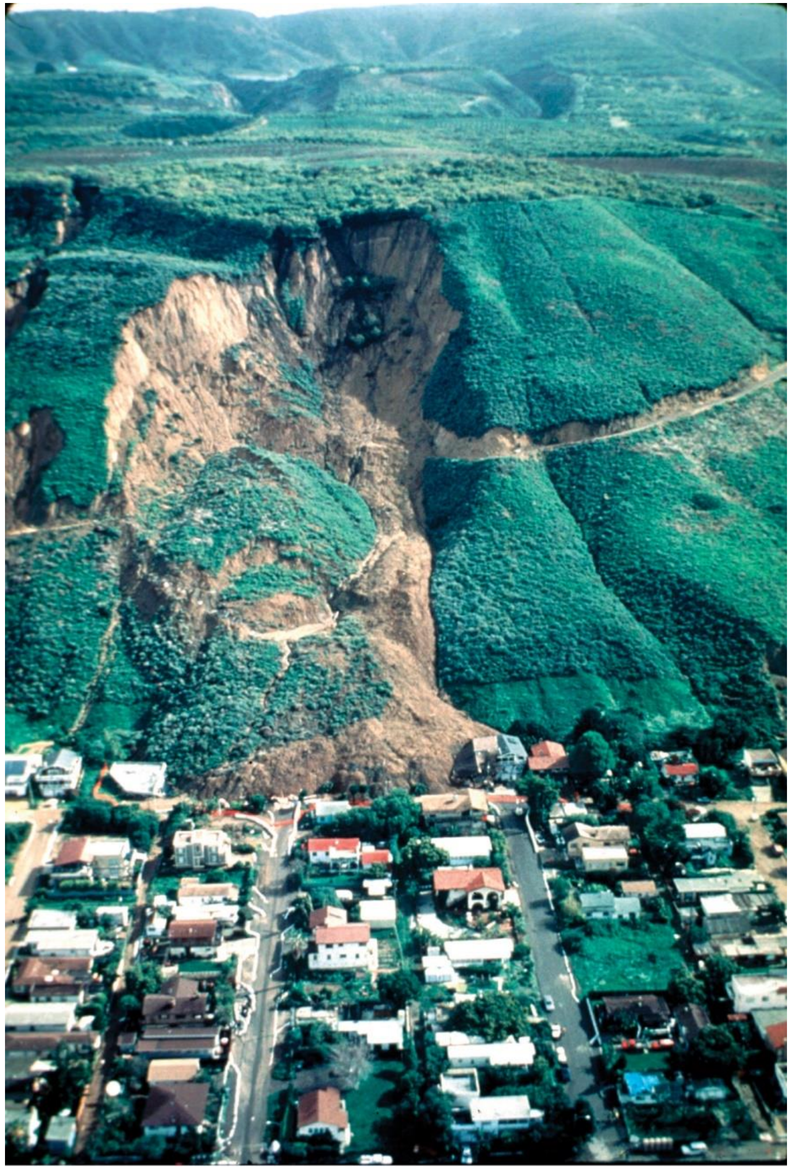
Deterministic factor of safety for design

- Several design charts for factor of safety (FS)
- Input values: slope geometry and best estimates of soil properties
- Deterministic FS is not enough for design: unable to account for uncertainties



Geotechnical uncertainties

- Inherent spatial variability of soil properties
- Scarcity of representative data
- Changing environmental conditions
- Unexpected failure mechanisms
- Model uncertainty
- Human error in design and construction



Reluctance in adopting probabilistic analysis

- Limited engineers' training in probability theory
- Engineers are more comfortable with deterministic analysis
- Common misconception that probabilistic analysis requires significantly more data, time, and effort
- Software availability (spatial variability)
- Probabilistic analysis is not a replacement for deterministic analysis

Coefficient of Variation

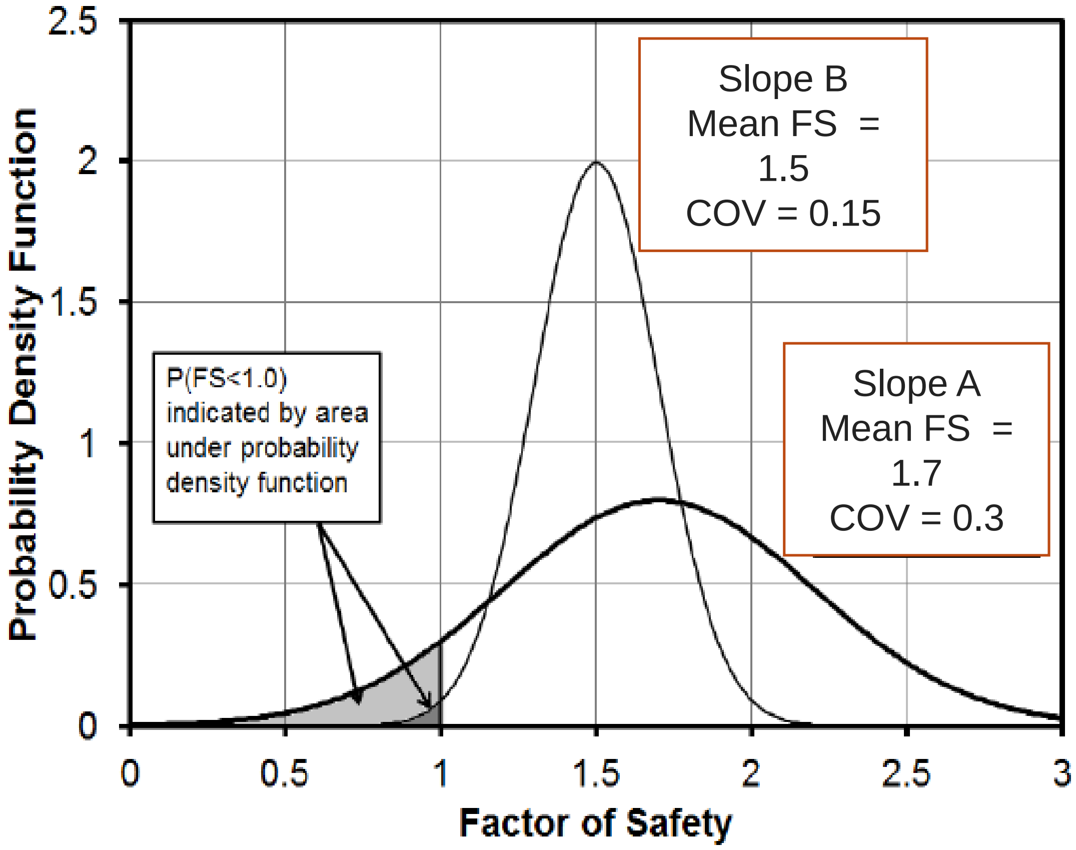
- Coefficient of variation (COV) shows the level of uncertainty in input parameters
- COV = standard deviation / mean
- e.g. COV of s_u is 10-55%

Test type	Property	Soil Type	Mean	COV(%)
Lab Strength	s_u(UC)	Clay	10 – 400 kN/m^2	20 – 55
	s_u(UU)	Clay	10 – 350 kN/m^2	10 – 30
	s_u(CIUC)	Clay	150 – 700 kN/m^2	20 – 40
	$\bar{\phi}$	Clay and sand	20 – 40°	5 – 15
CPT	q_T	Clay	0.5 – 2.5 MN/m^2	<20
	q_c	Clay	0.5 – 2.0 MN/m^2	20 – 40
	q_c	Sand	0.5 – 30.0 MN/m^2	20 – 60
VST	s_u(VST)	Clay	5 – 400 kN/m^2	10 – 40
SPT	N	Clay and sand	10 – 70 blows/ft	25 – 50
DMT	A	Clay	100 – 450 kN/m^2	10 – 35
	A	Sand	60 – 1300 kN/m^2	20 – 50
	B	Clay	500 – 880 kN/m^2	10 – 35
	B	Sand	350 – 2400 kN/m^2	20 – 50
	I_D	Sand	1 – 8	20 – 60
	K_D	Sand	2 – 30	20 – 60
	E_D	Sand	10 – 50 MN/m^2	15 – 65
PMT	p_L	Clay	400 – 2800 kN/m^2	10 – 35
	p_L	Sand	1600 – 3500 kN/m^2	20 – 50
	E_{PMT}	Sand	5 – 15 MN/m^2	15 – 65
Lab index	w_n	Clay and silt	13 – 100%	8 – 30
	w_L	Clay and silt	30 – 90%	6 – 30
	w_p	Clay and silt	15 – 25%	6 – 30
	PI	Clay and silt	10 – 40%	—[a]
	LI	Clay and silt	10%	—[a]
	γ, γ_d	Clay and silt	13 – 20 kN/m^3	<10
	D_r	Sand	30 – 70%	10 – 40; 50 – 70[b]

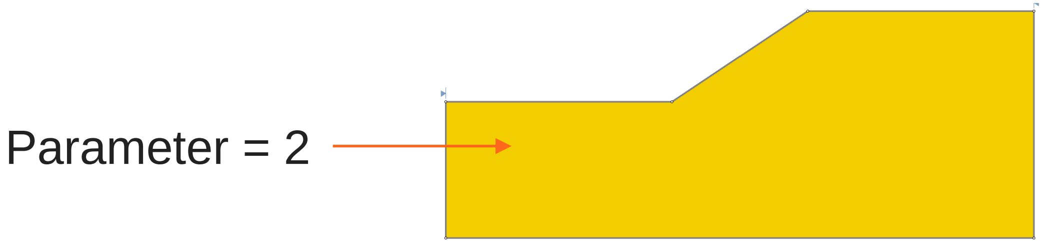
[a]COV = (3-12%)/mean
[b]The first range of values gives the total variability for the direct method of determination, and the second range of values the total variability for the indirect determination using SPT values

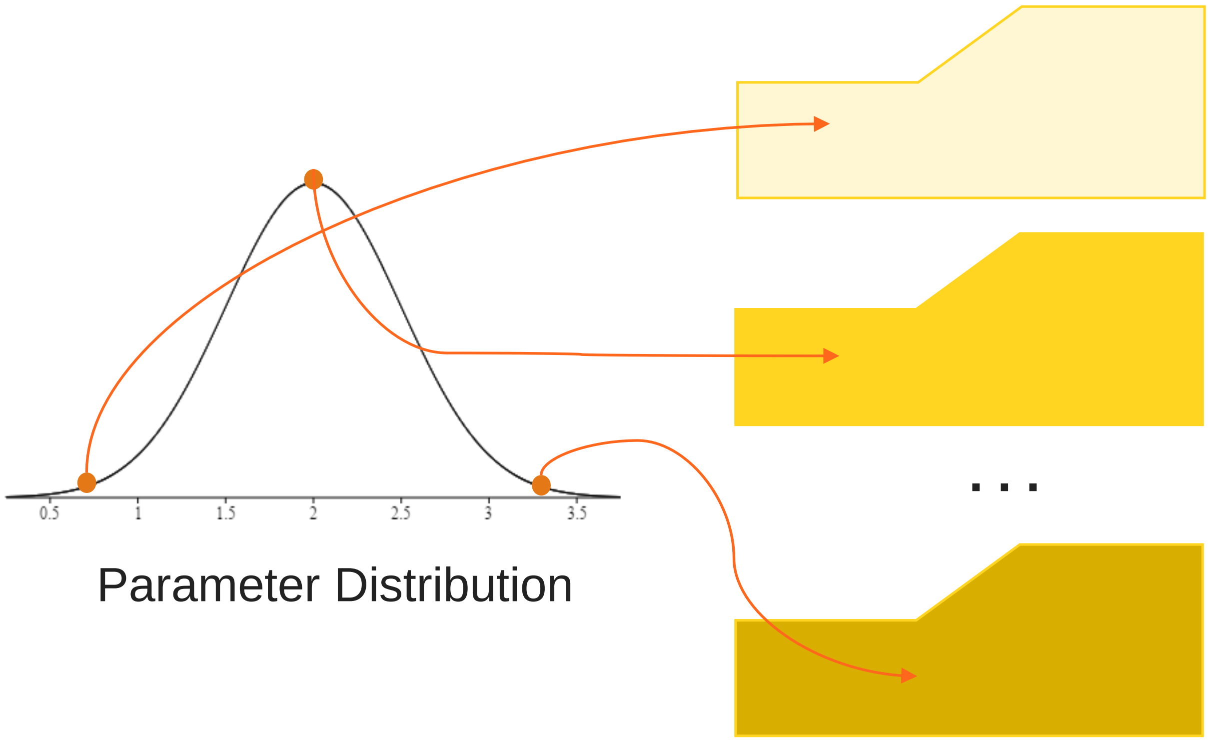
Importance of COV

- Probability of failure (PF) is the area under the distribution of FS for FS < 1
- Slope A with a higher Mean FS is less safe (has higher PF) because it has higher COV



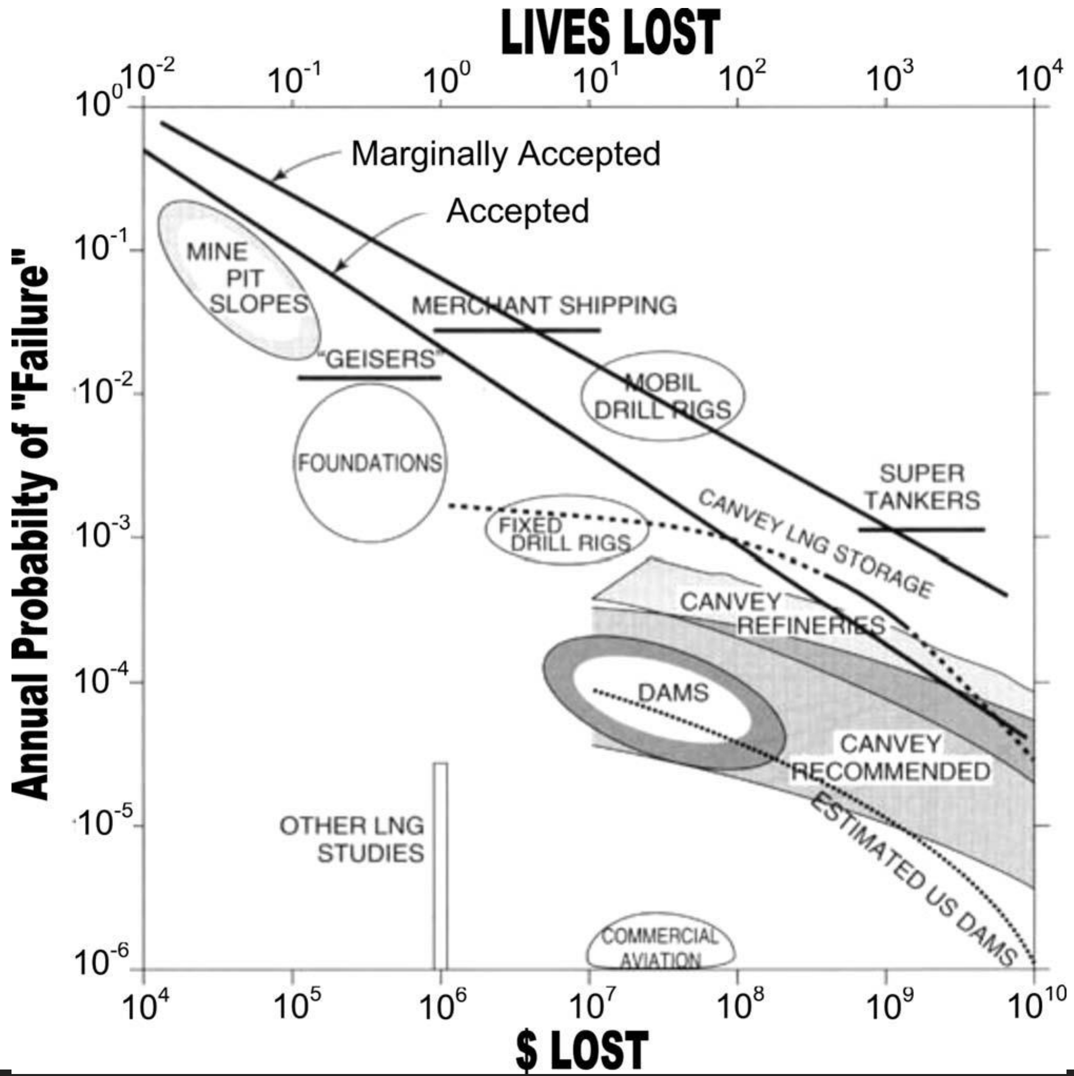
Deterministic vs. Probabilistic





$$Probability\ of\ Failiure = \frac{\#\ of\ simulations\ with\ FS < 1}{Total\ \#\ of\ simulations}$$

Acceptable p_f values



Silva et al. (2008)

Simple unreinforced slopes

Stability charts for cohesive slopes (su)

Probability theory and Taylor's equation were used to develop a unique equation

Probability Theory: → $$P_f = p[Z < a] = \Phi\left(\frac{\ln a - \mu_{\ln Z}}{\sigma_{\ln Z}}\right)$$

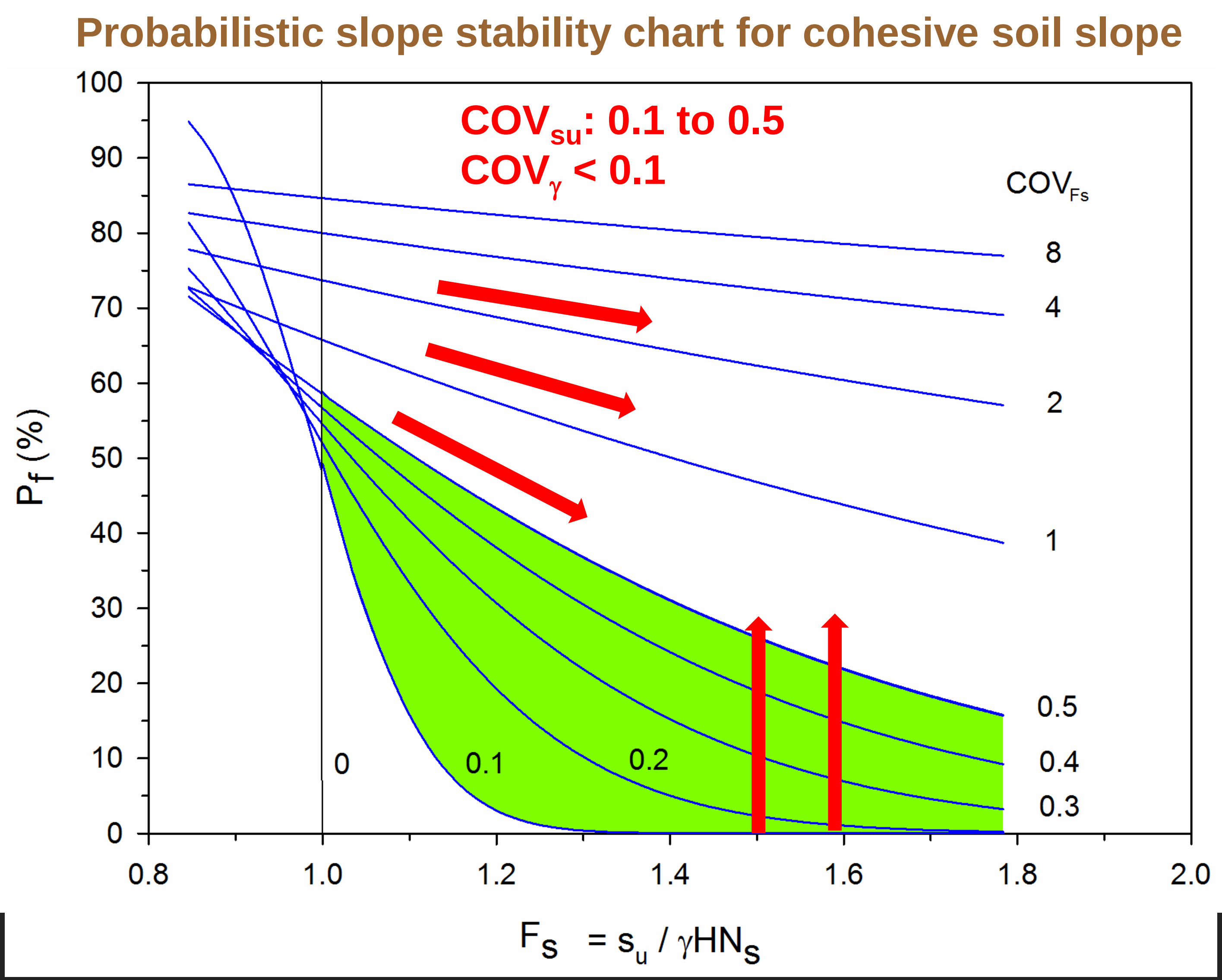
Taylor's equation: → $$F_s = \frac{s_u}{\gamma H N_s}$$

Probability of failure: → $$P_f = p[F_s < 1] = \Phi\left(\frac{\ln\left(\sqrt{\frac{1+COV_{su}^2}{1+COV_{\gamma}^2}} / \bar{F}_S\right)}{\sqrt{\ln\left[\left(1+COV_{su}^2\right)\left(1+COV_{\gamma}^2\right)\right]}}\right)$$

Advantage: **Probability of failure can be calculated directly using F_s and spread in s_u and γ**

Javankhoshdel and Bathurst (2014)

Stability charts for cohesive slopes (su)



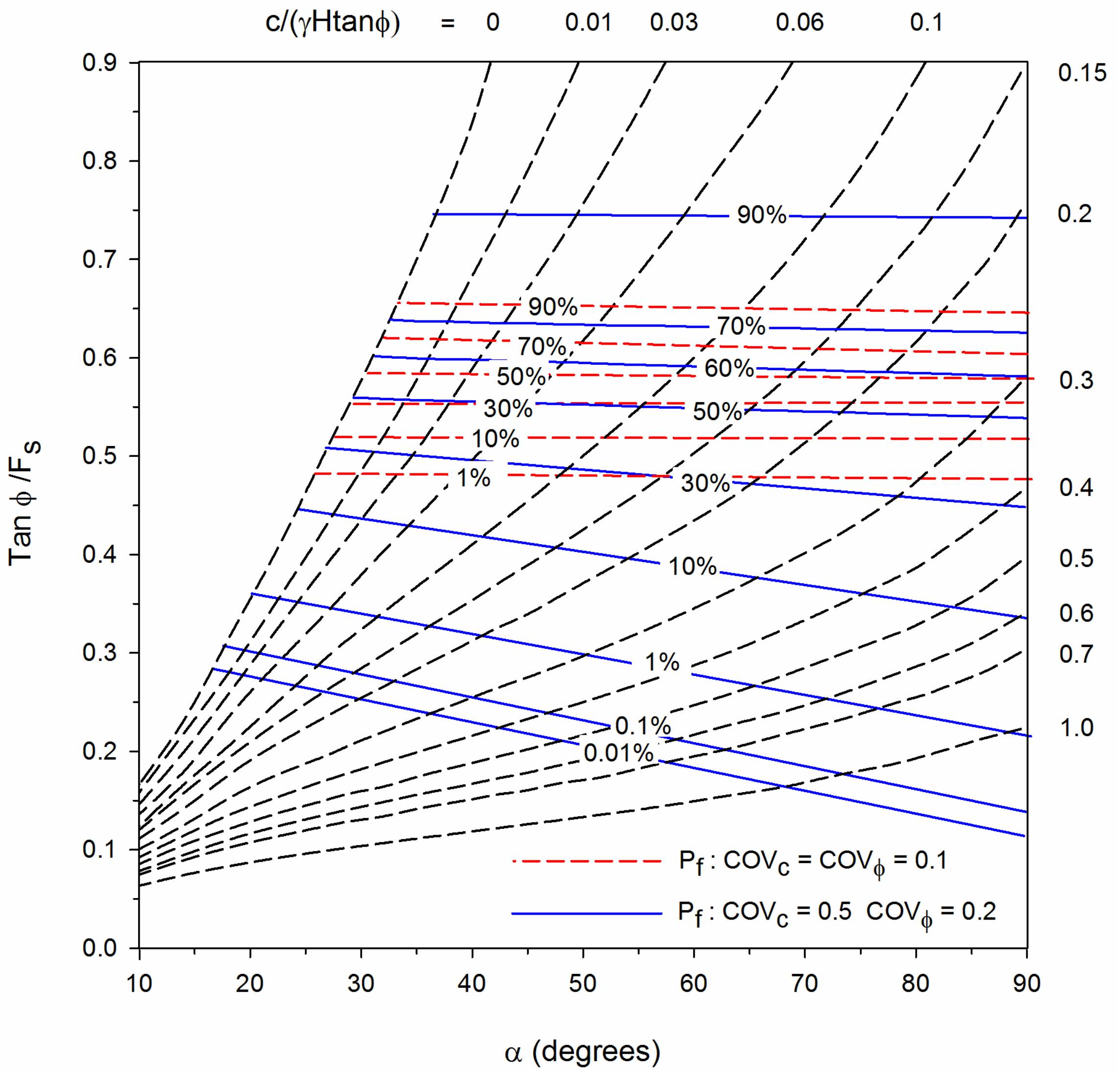
Javankhoshdel and Bathurst (2014)

Stability charts for cohesive-frictional slopes (c-ϕ)

- **Monte Carlo simulation**
- **6 different charts for:**

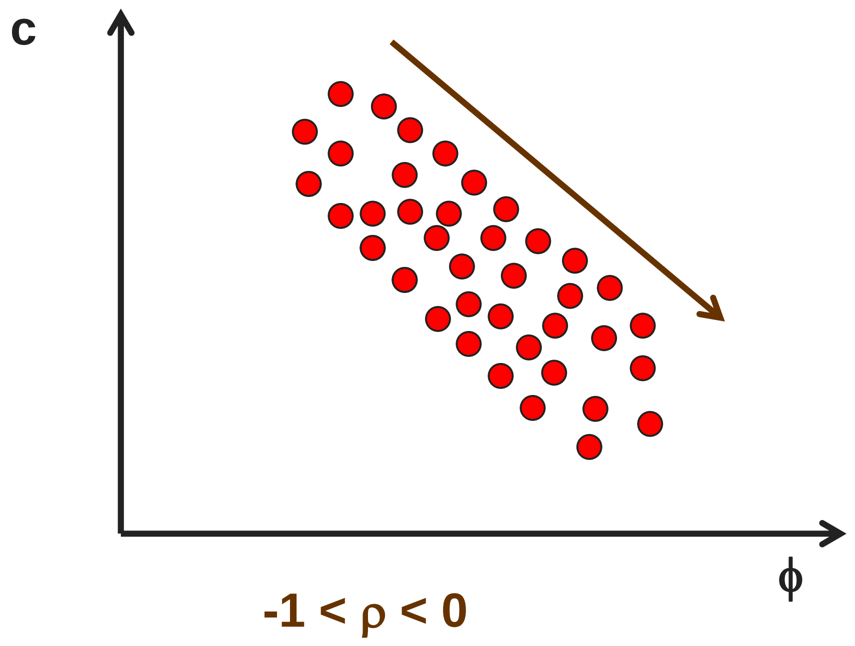
  $20^\circ \leq \phi \leq 45^\circ$

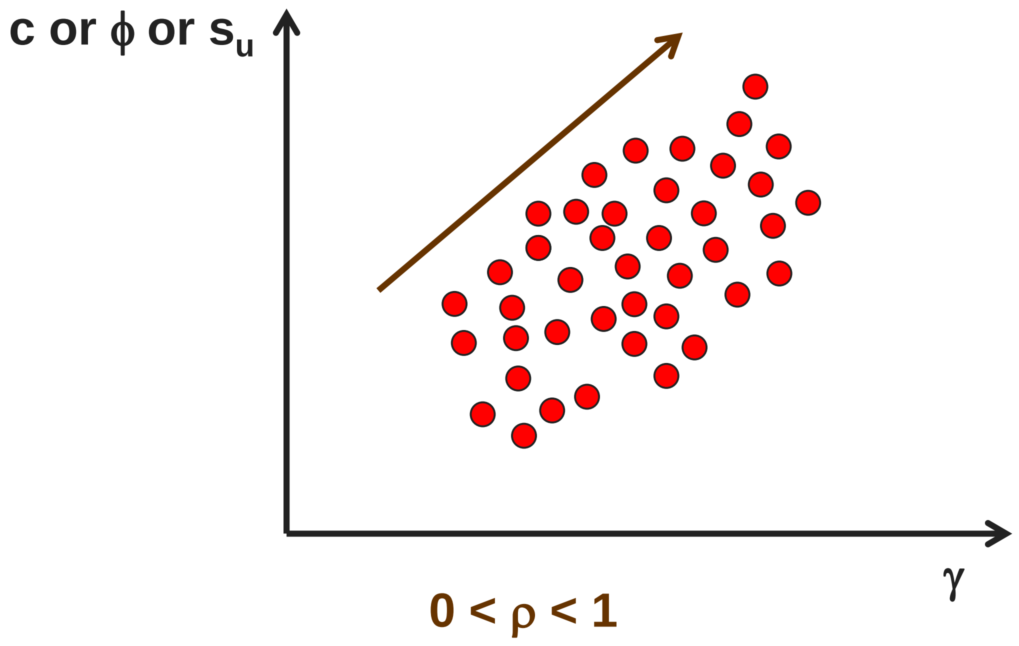
- **F_s and P_f in one chart**
- **COV_c = 0.5 and 0.1**
  COV_ϕ = 0.2 and 0.1



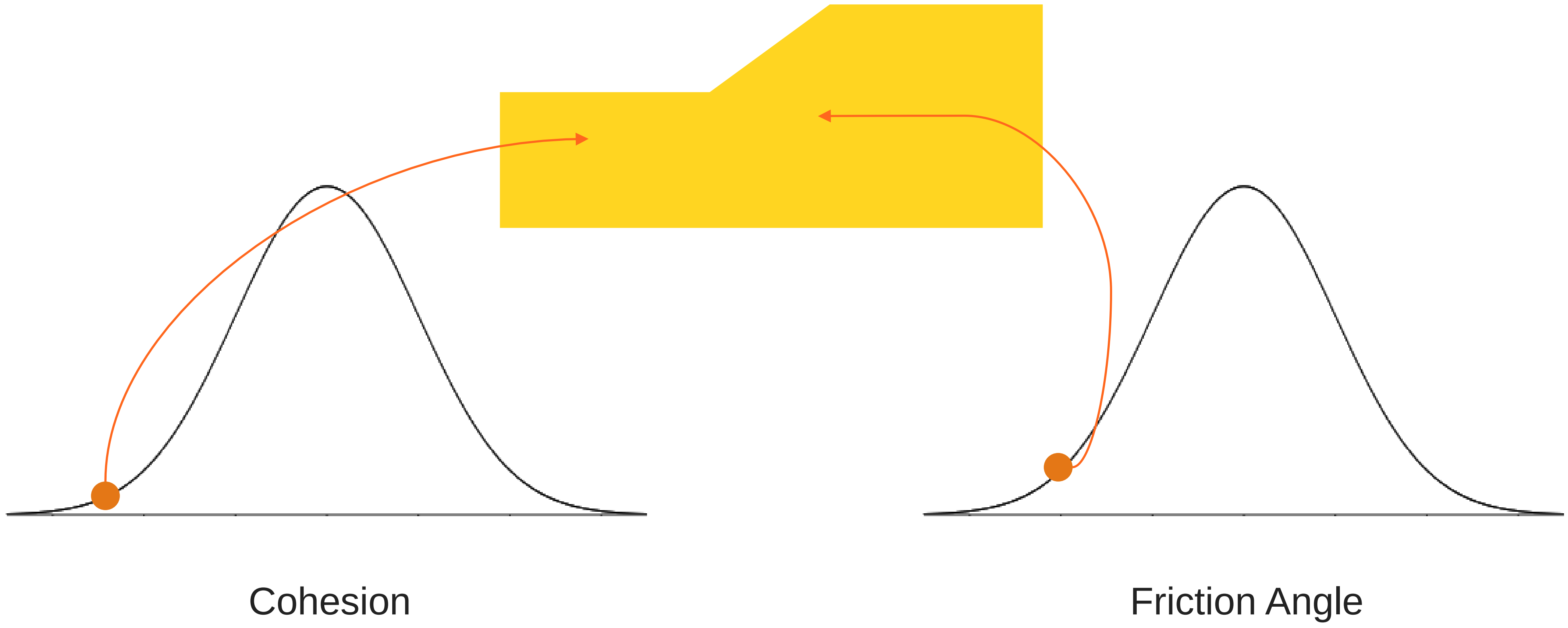
High values of Probability of failure

Cross-correlation (ρ): Considered to incorporate the dependency between input parameters

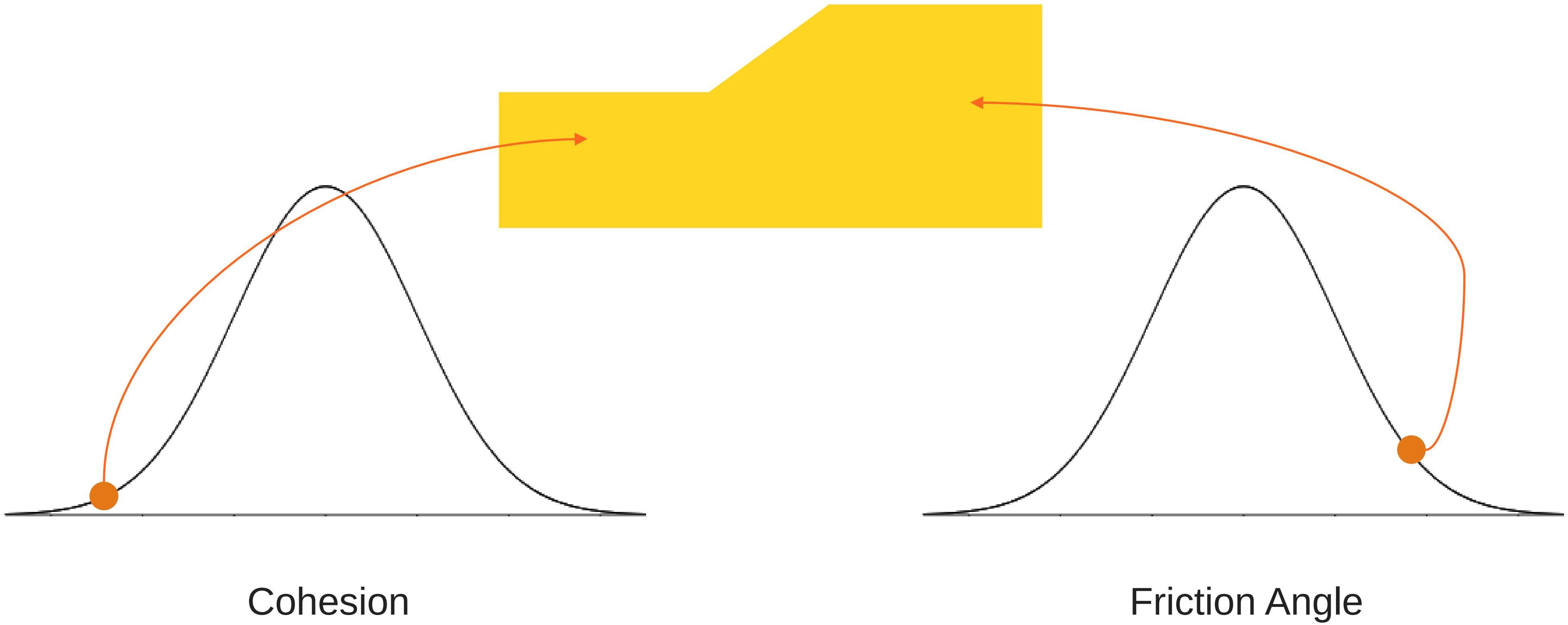




Sampling of parameters



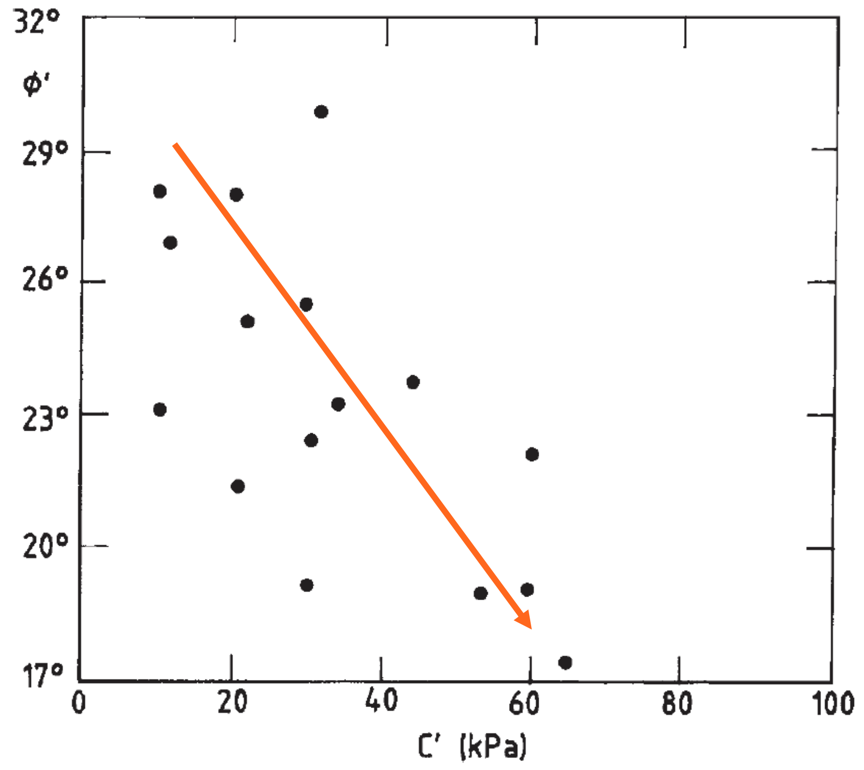
Sampling of parameters



Cross-correlation between soil parameters

- Cross-correlation (ρ): Considered to incorporate the dependency between input parameters
- -1.0 ≤ ρ ≤ 1.0
- A negative ρ is recommended between c and ϕ
- A positive ρ is recommended between c and γ; ϕ and γ

Fig. 1. Values of drained cohesion c' and friction angle ϕ' for samples of Matera Blue Clay.



The effect of cross-correlation

Combination 1: Unrealistic ρ

c & ϕ = +0.5

c & γ = -0.5

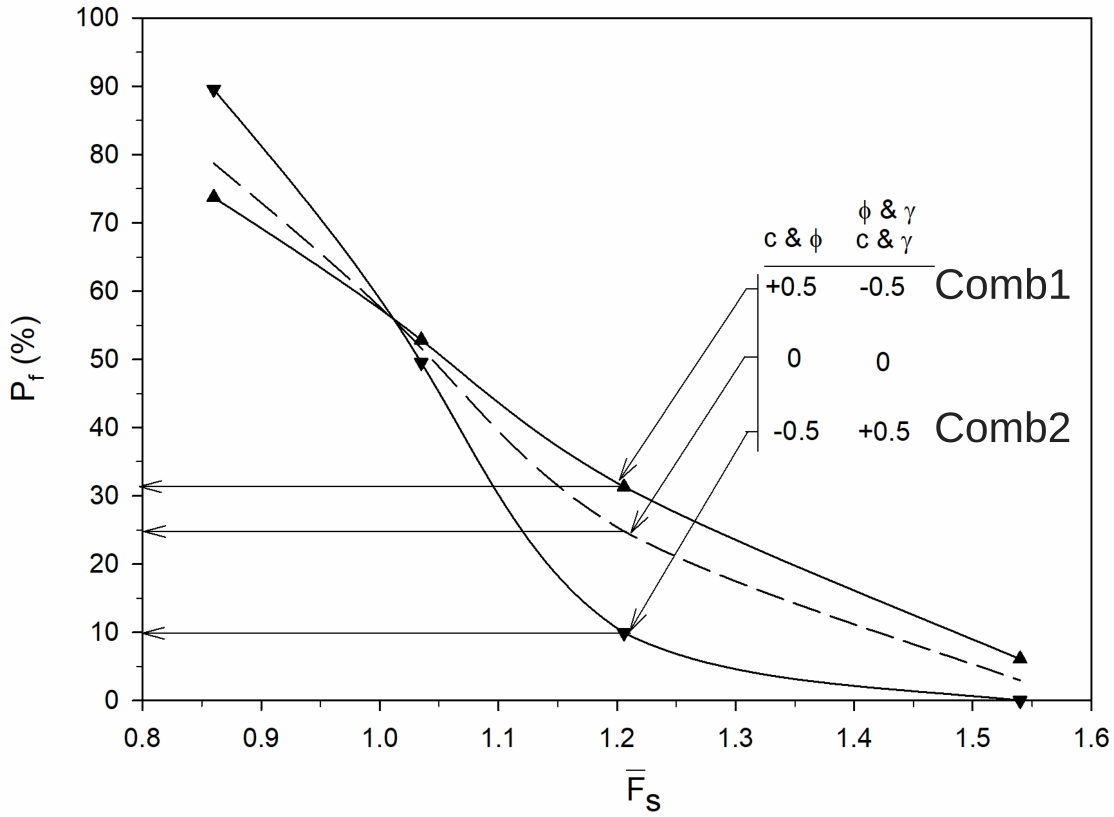
ϕ & γ = -0.5

Combination 2: Realistic ρ

c & ϕ = -0.5

c & γ = +0.5

ϕ & γ = +0.5



Cross-correlation between s_u and γ

Probability of failure: $\xrightarrow{\rho = 0}$ $$P_f = p[F_s < 1] = \Phi\left(\frac{\ln\left(\sqrt{\dfrac{1+COV_{su}^2}{1+COV_\gamma^2}}\,/\,\bar{F}_S\right)}{\sqrt{\ln\left[\left(1+COV_{su}^2\right)\left(1+COV_\gamma^2\right)\right]}}\right)$$

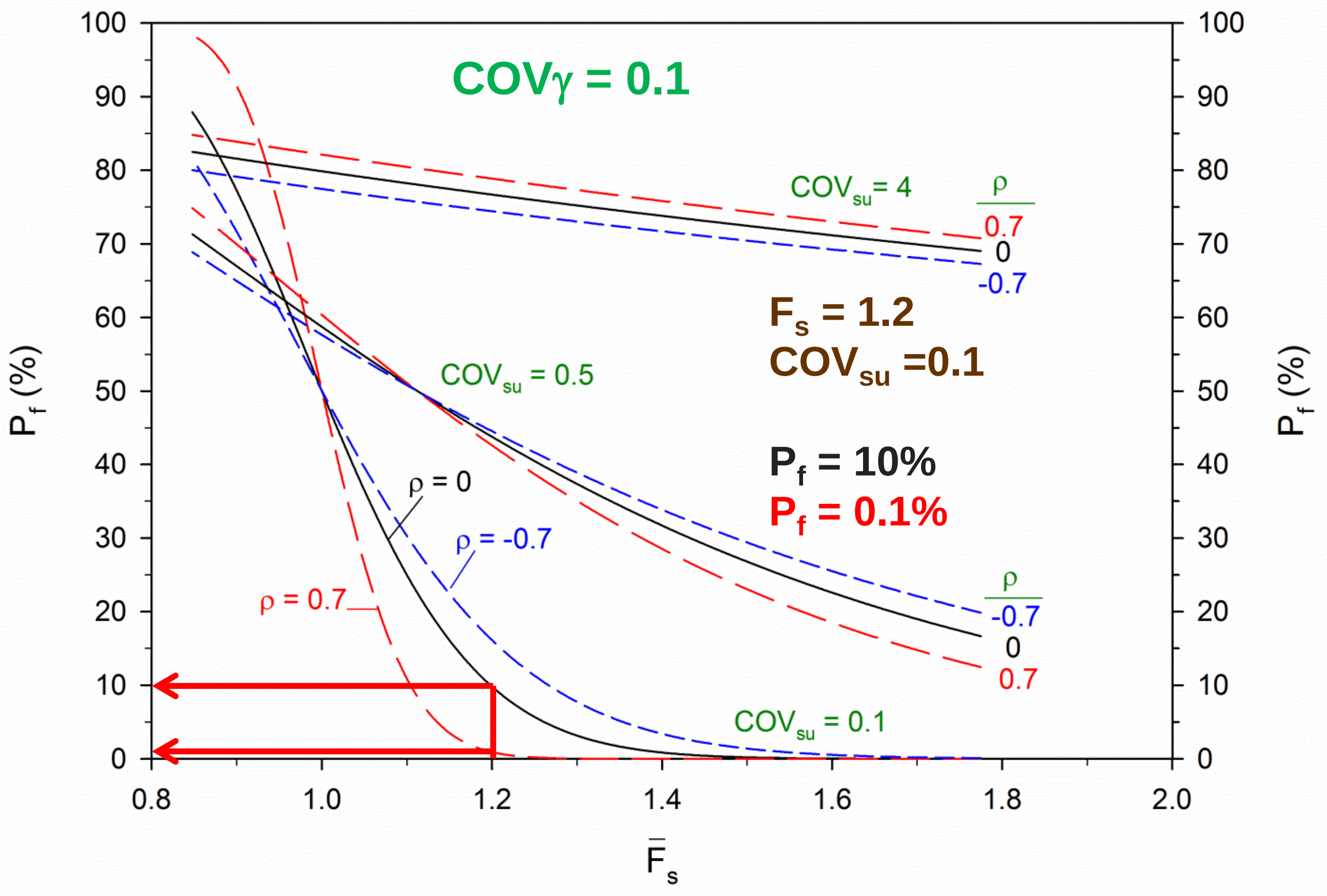
Javankhoshdel and Bathurst (2014)

Probability of failure: $\xrightarrow{\rho \neq 0}$ $$P_f = p[F_s < 1] = \Phi\left(\frac{\ln\left(\sqrt{\dfrac{1+COV_{su}^2}{1+COV_\gamma^2}}\,/\,\bar{F}_S\right)}{\sqrt{\ln((1+COV_{su}^2)(1+COV_\gamma^2)/(1+\rho COV_{su}COV_\gamma)^2)}}\right)$$

Javankhoshdel and Bathurst (2016a)

Cross-correlation between s_u and γ

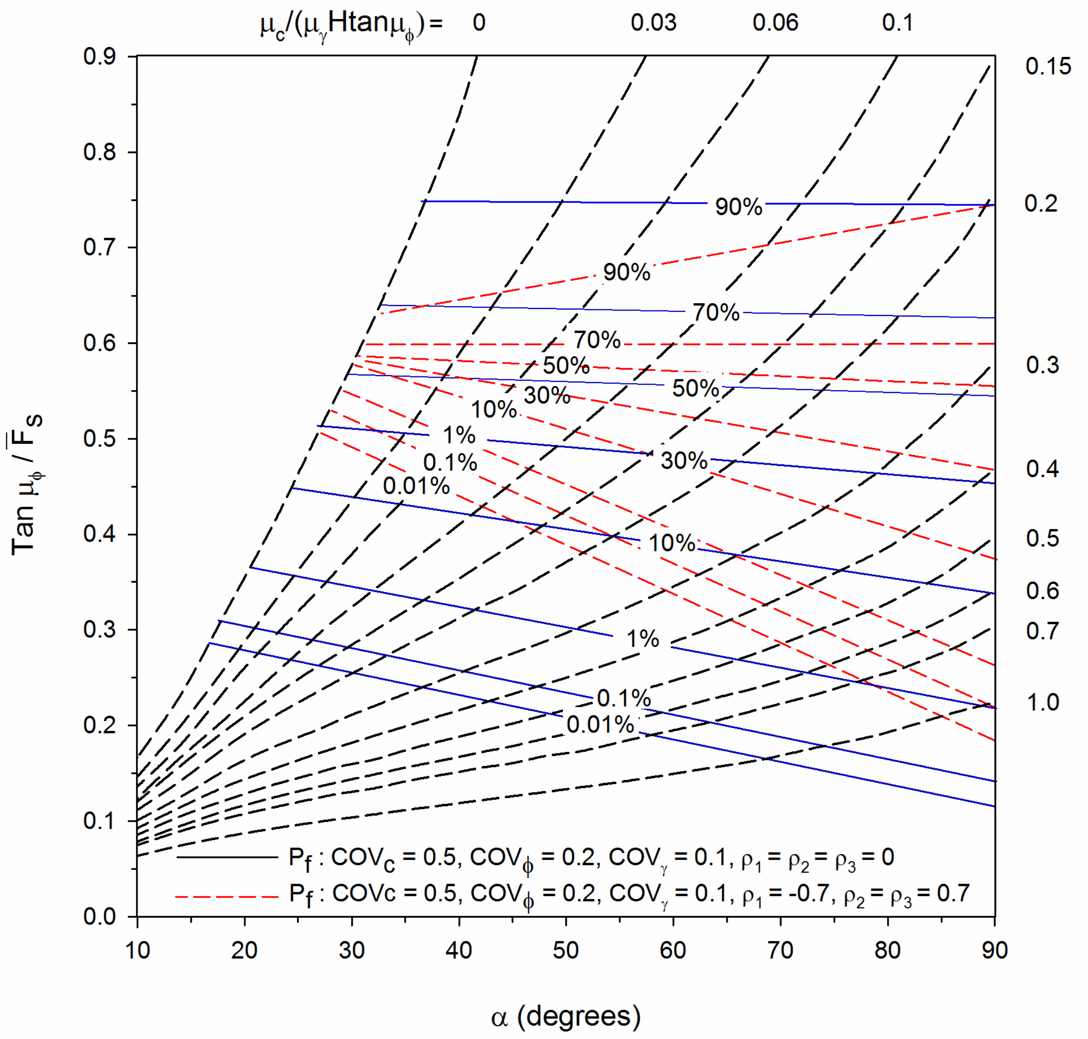
Influence of cross-correlation between s_u and γ on probability of failure



Javankhoshdel and Bathurst (2016a)

Stability chart for cohesive-frictional soils (c-ϕ)

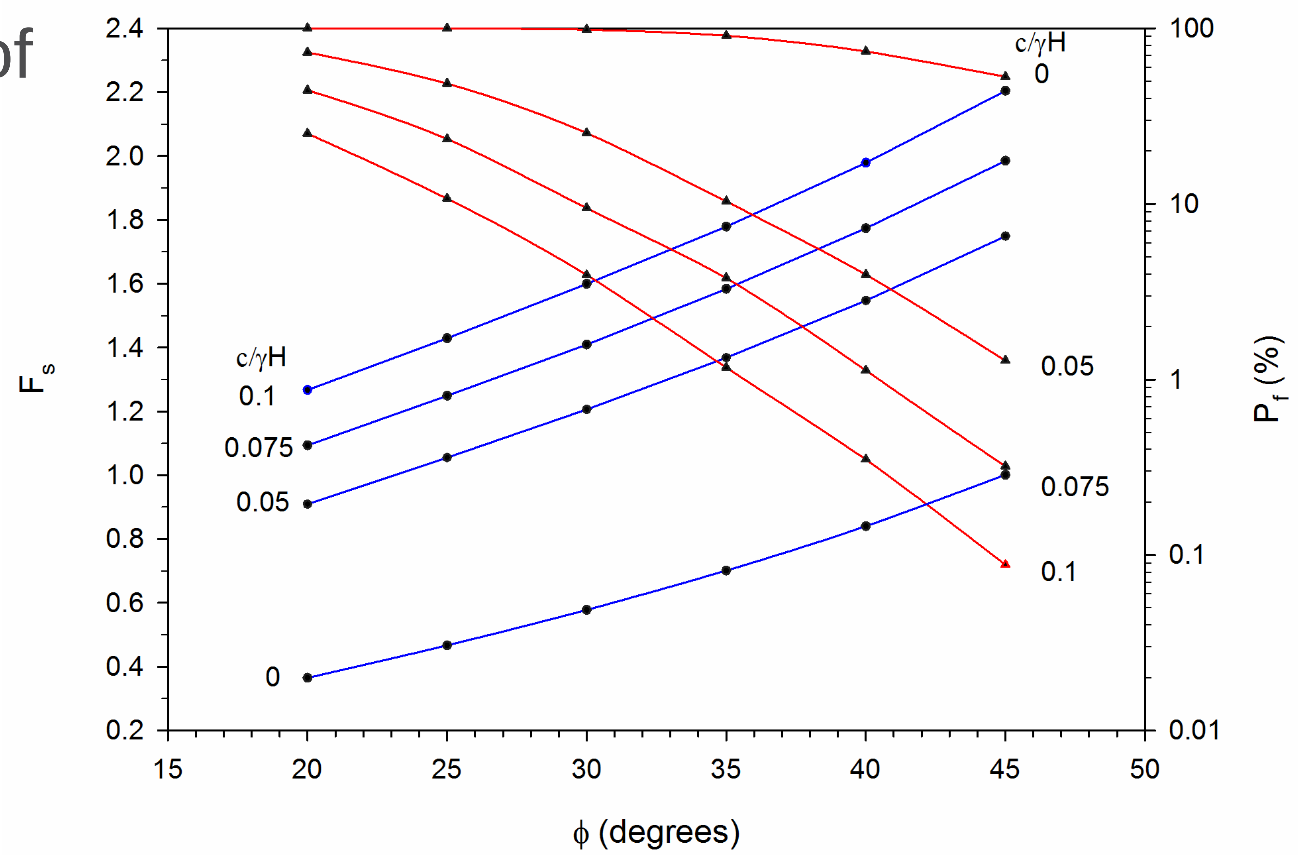
- **Monte Carlo simulation**
- **6 different charts for:**
  20˚ ≤ ϕ ≤ 45˚
- **F_s and P_f in one chart**
- **$COV_c = 0.5$**
  $COV_\phi = 0.2$
  $COV_\gamma = 0.1$
- **Maximum practical values of cross-correlation**
  $\rho_1 = -0.7$ and $\rho_2 = \rho_3 = +0.7$



Javankhoshdel and Bathurst (2016a)

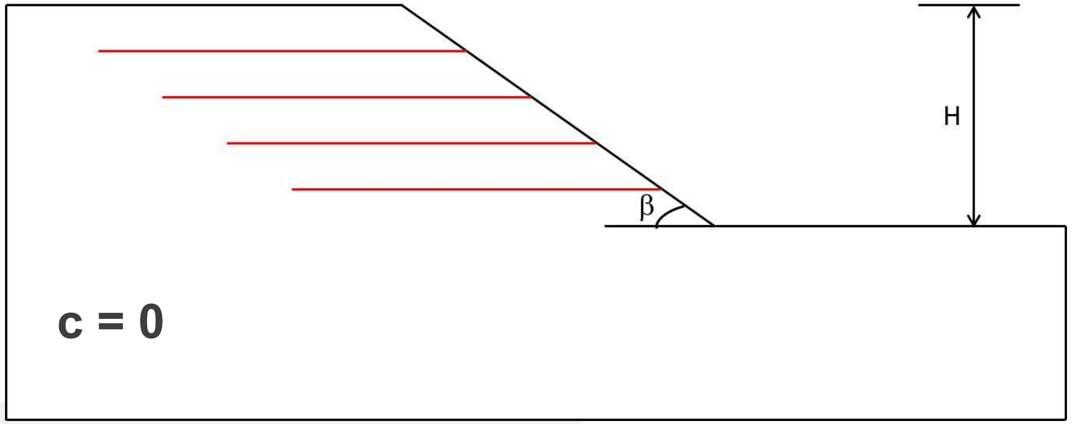
Stability chart for cohesive-frictional soils(c-ϕ)

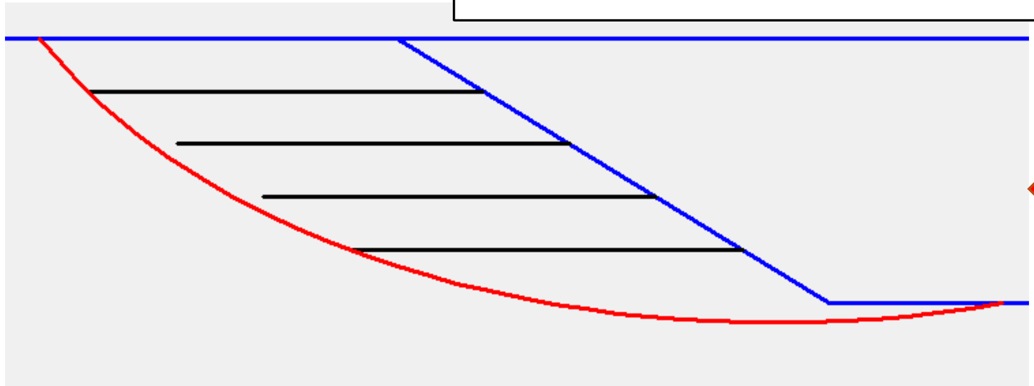
- Example for the slope angle of 45 degrees
- The factor of safety can be linearly interpolated with friction angle to sufficient practical accuracy
- The logarithm of the corresponding probability of failure also varies smoothly with friction angle



Simple reinforced slopes

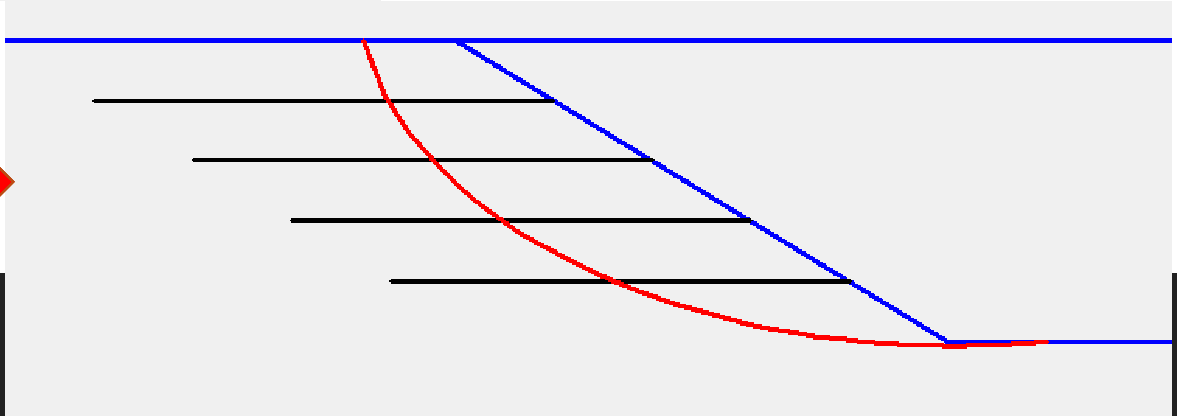
Deterministic analysis of reinforced slopes



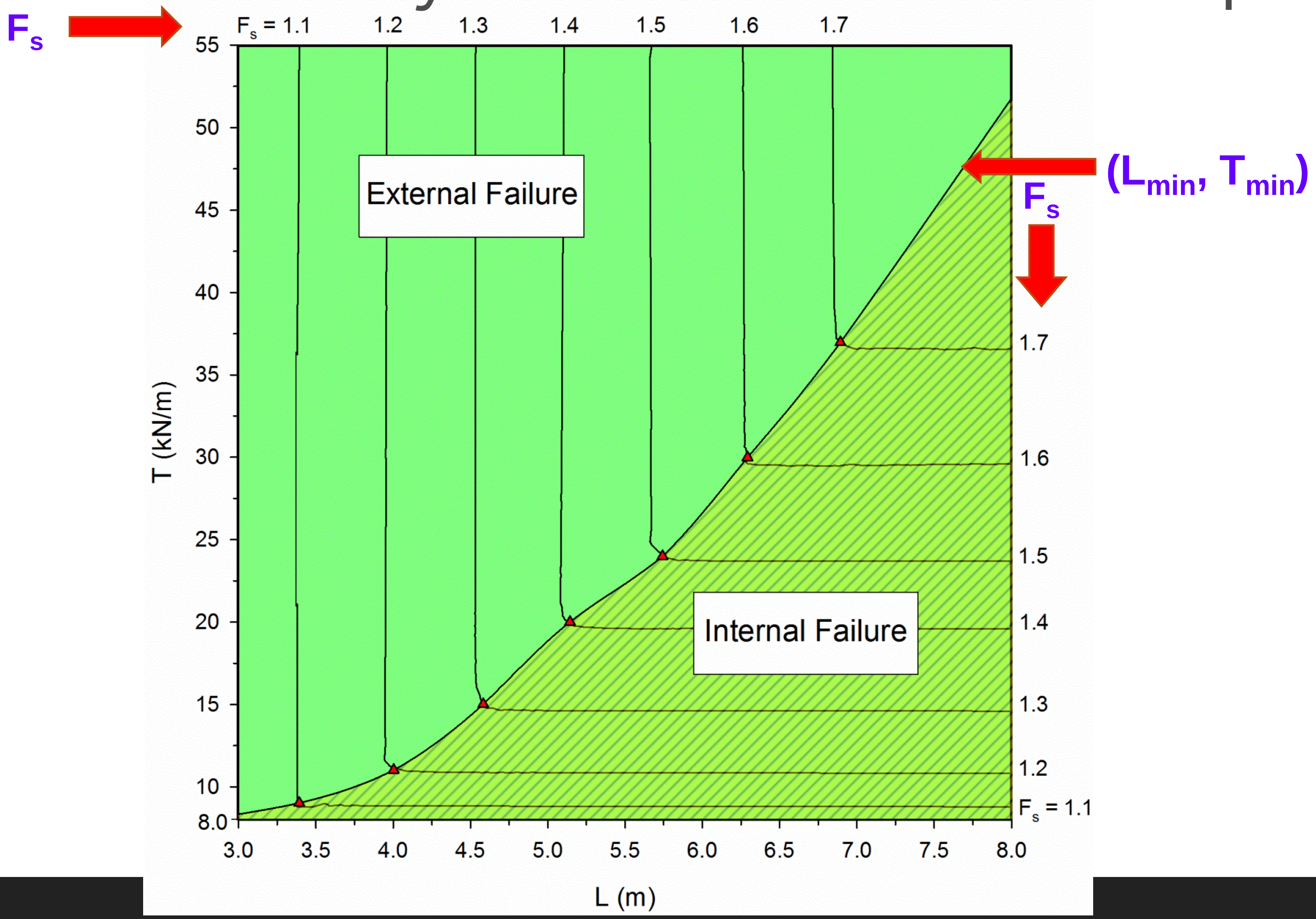


External failure

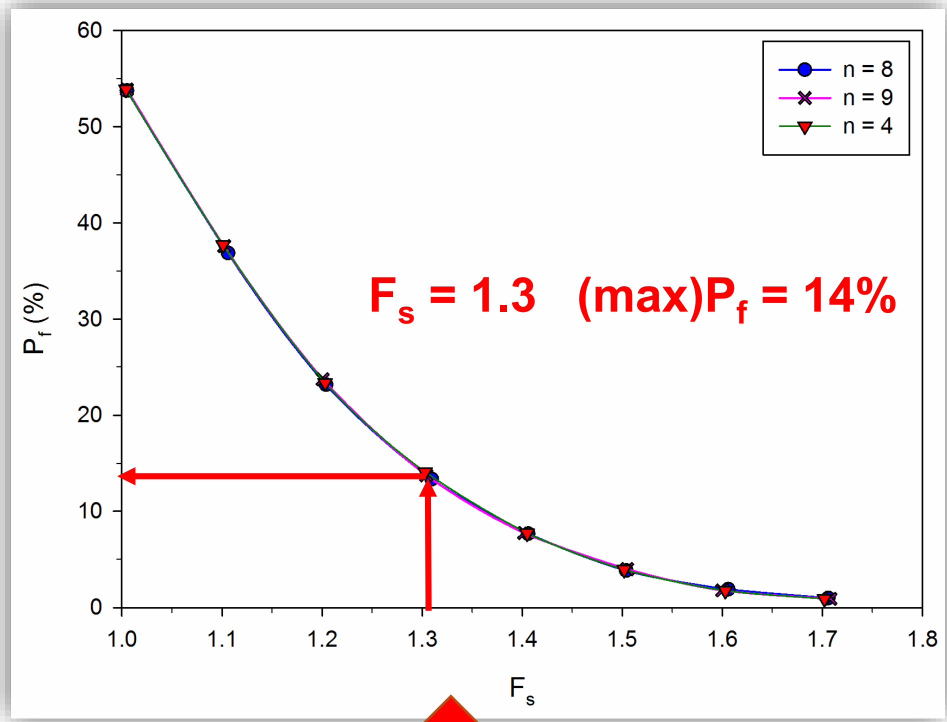
Internal failure



Deterministic analysis of reinforced slopes



Probabilistic analysis of reinforced slopes



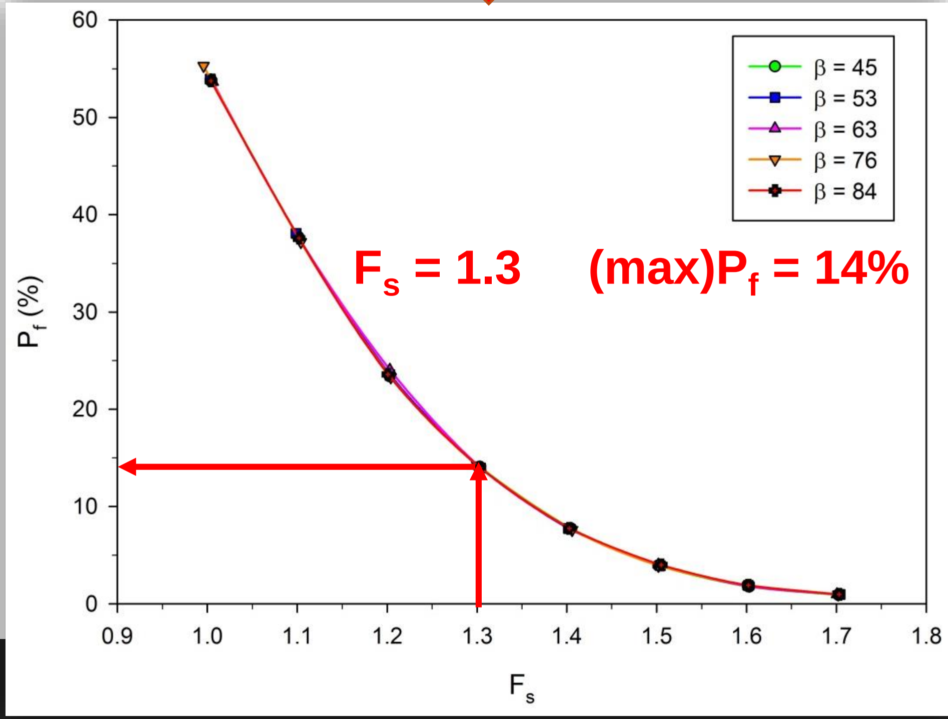
Effect of number of reinforcement layers

External failure

(PRSS software)

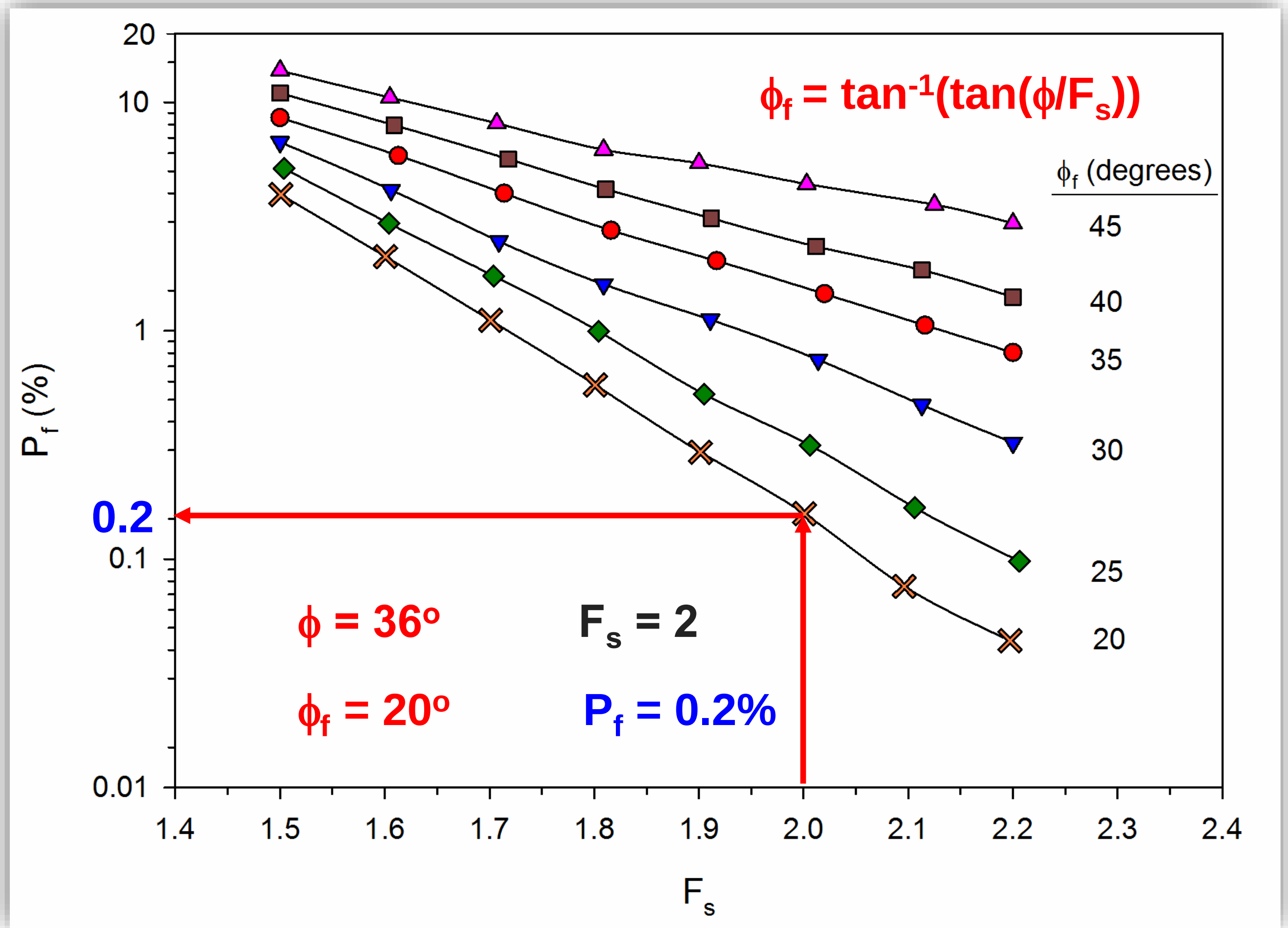
Effect of slope angle

COV_ϕ = 0.2
COV_γ = 0.1
COV_T = 0.15



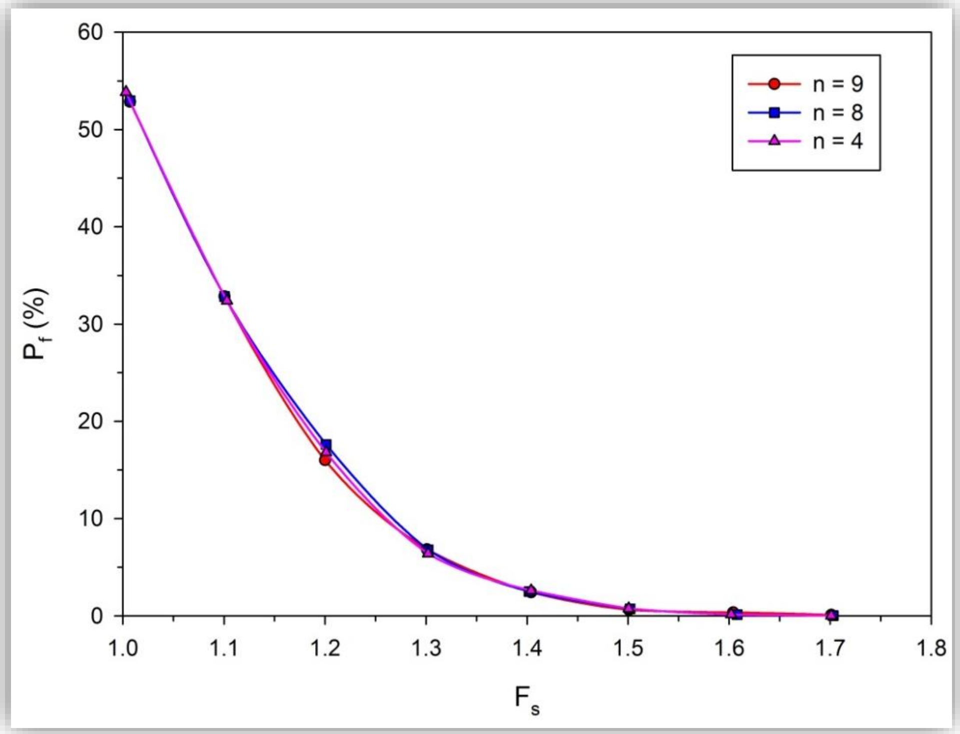
Probabilistic analysis of reinforced slopes

Probabilistic slope stability design chart for reinforced slopes with external failure



Javankhoshdel and Bathurst (2016b)

Probabilistic analysis of reinforced slopes

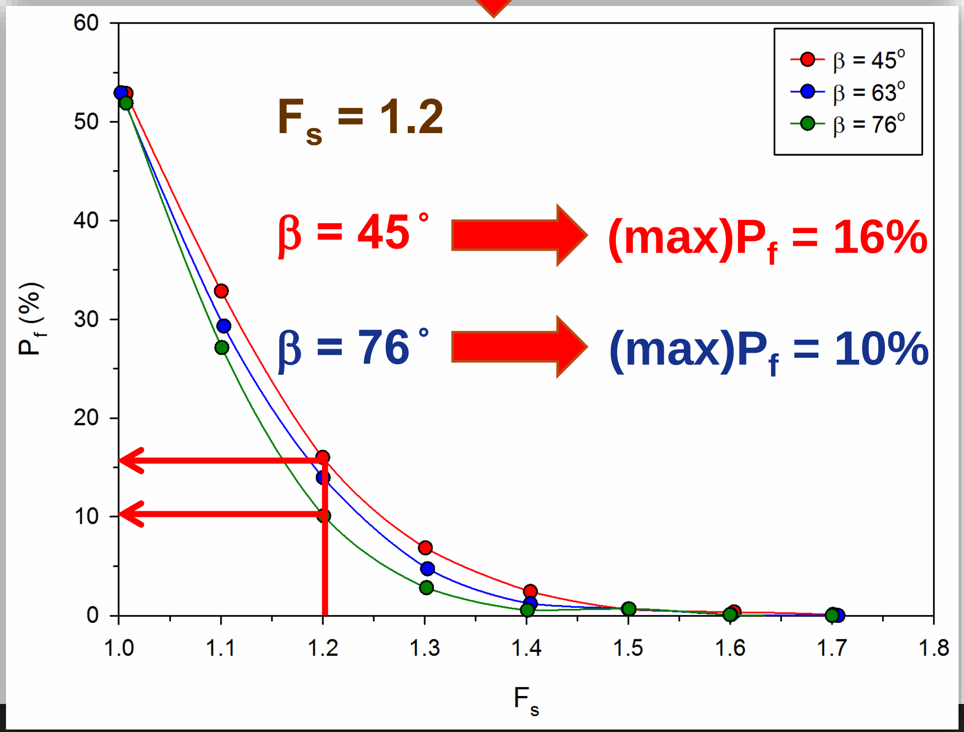


Effect of number of reinforcement layers

Internal failure

(PRSS software)

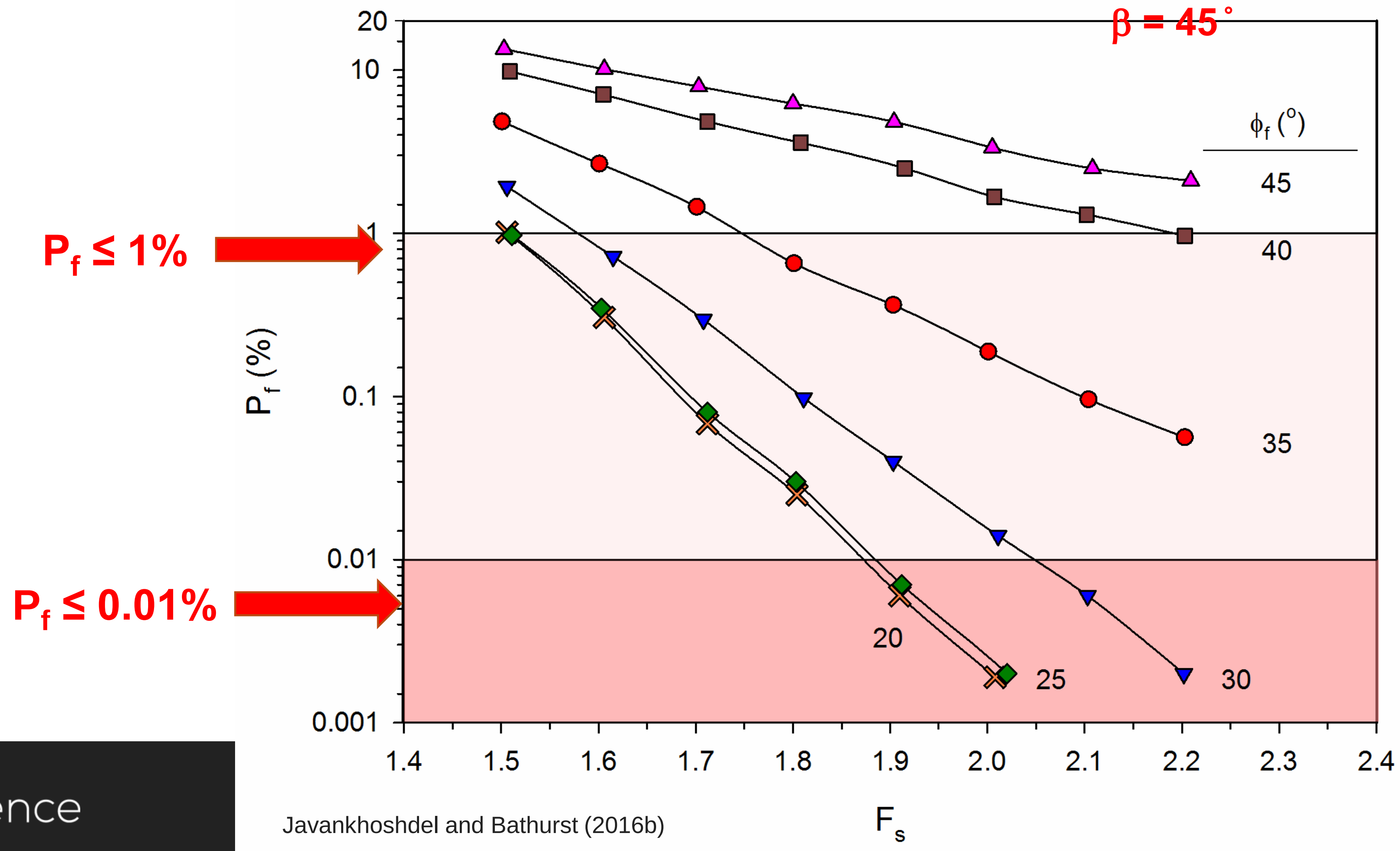
Effect of slope angle



Javankhoshdel and Bathurst (2016b)

Probabilistic analysis of reinforced slopes

Probabilistic slope stability design chart for reinforced slopes with internal failure



Javankhoshdel and Bathurst (2016b)

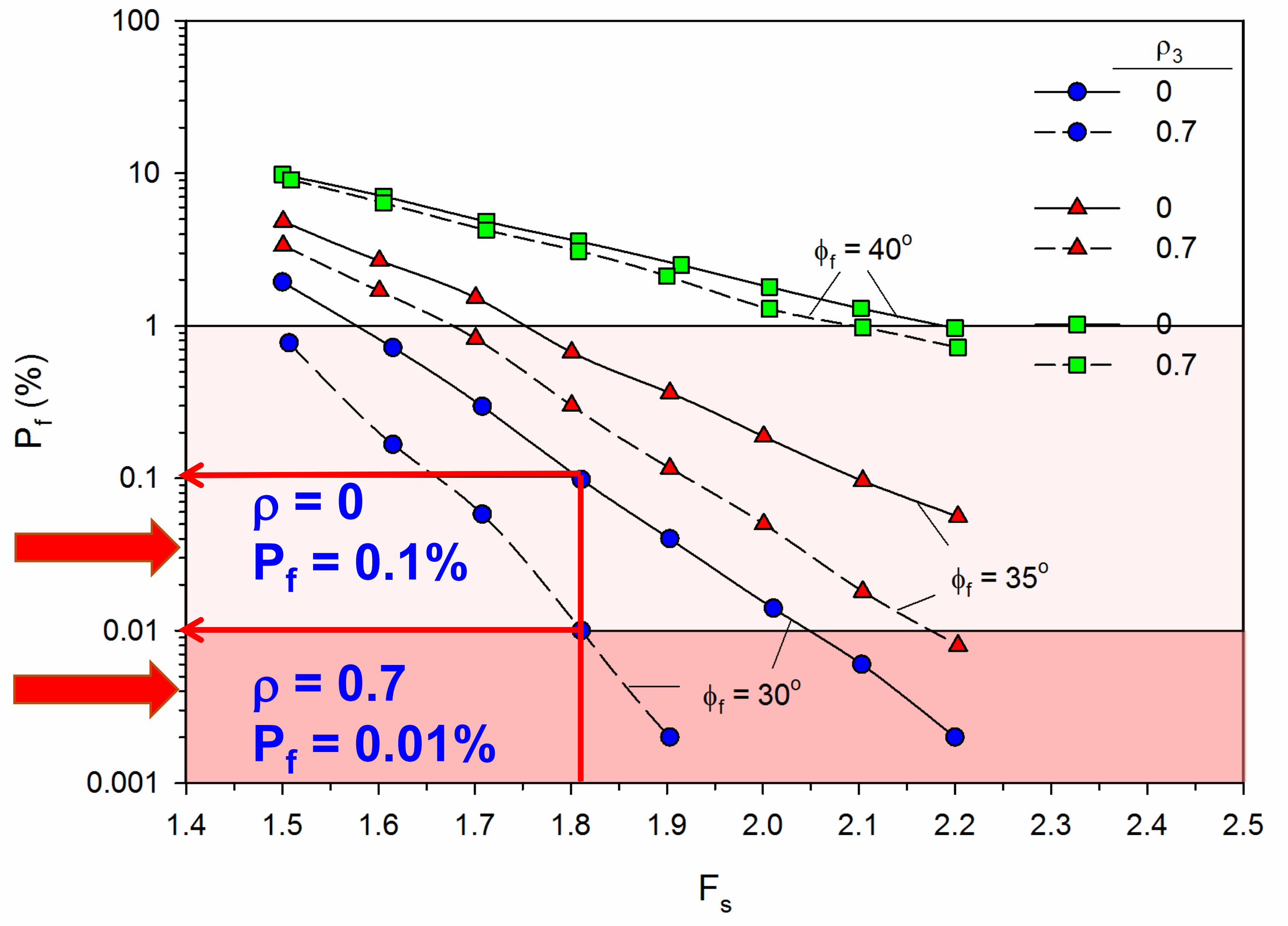
Cross-correlation between strength parameters

There is no cross-correlation effect for external failure : $F_s = \tan\phi/\tan\beta$

Influence of cross-correlation between ϕ and γ on probability of failure for internal failure

$P_f \leq 1\%$

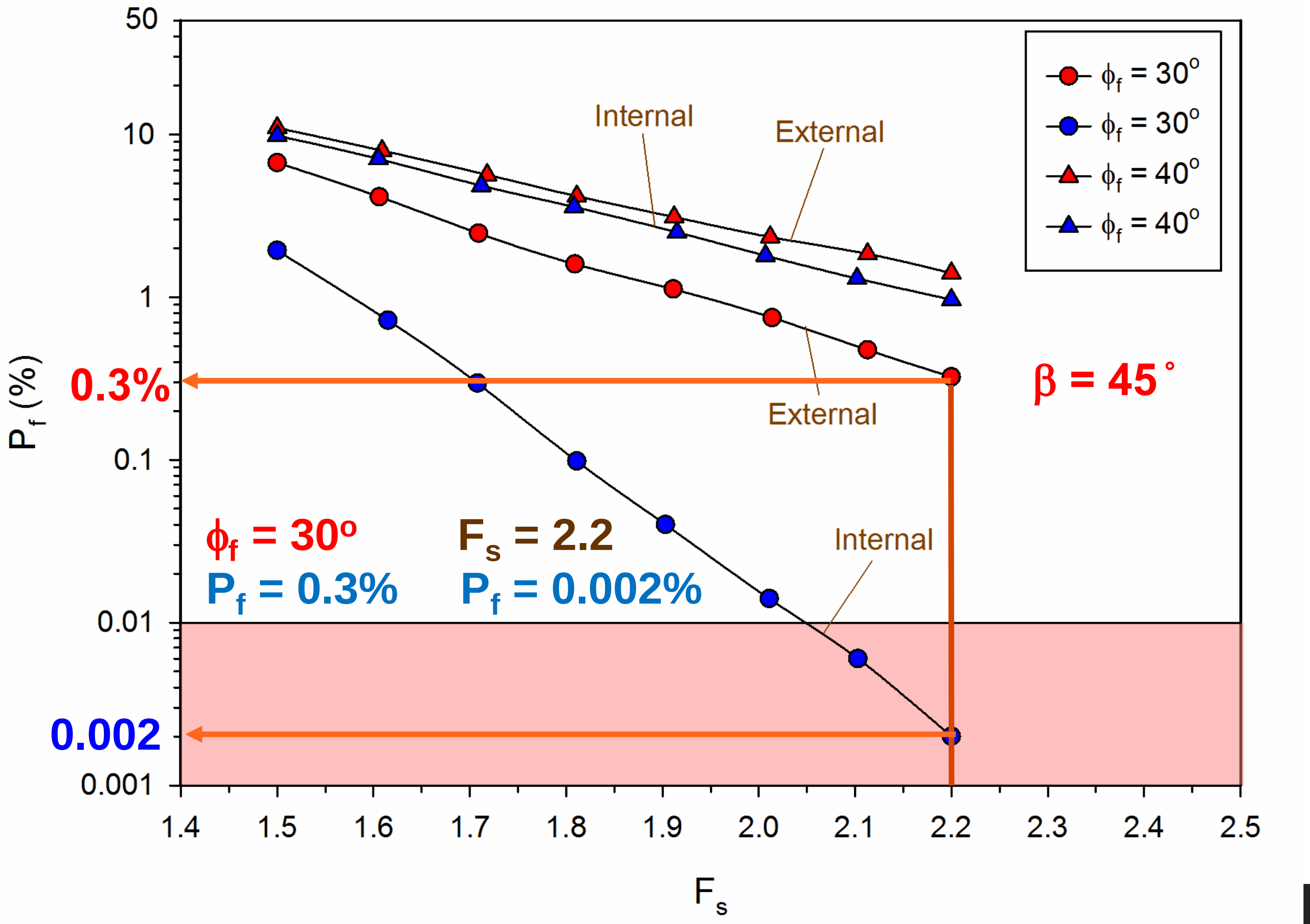
$P_f \leq 0.01\%$



Javankhoshdel and Bathurst (2016b)

Probabilistic analysis of reinforced slopes

Comparison between the probability of failure of external and internal failure mechanism

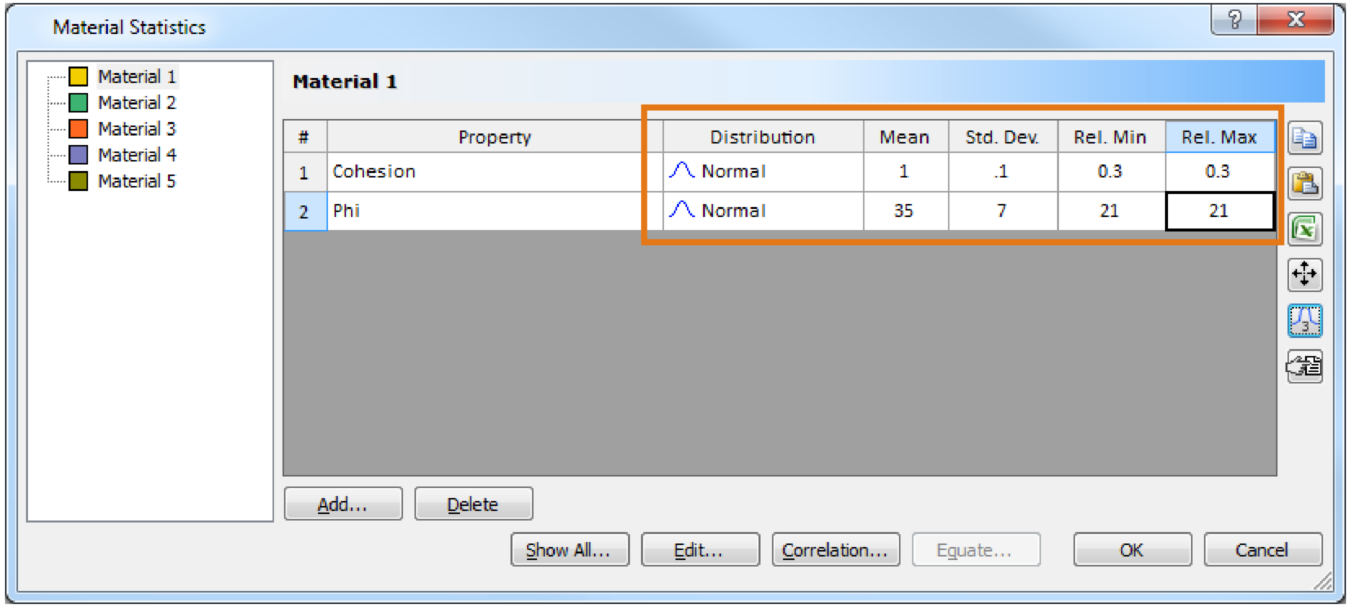


Simple Probabilistic Slope Stability Analysis in *Slide2*

Section 2

Probabilistic Slope Stability

Assign probability distributions to properties → Generate n samples for each property → Calculate FS for each simulation → Calculate PF



Lower bound = Mean – Rel. Min
Upper bound = Mean + Rel. Max

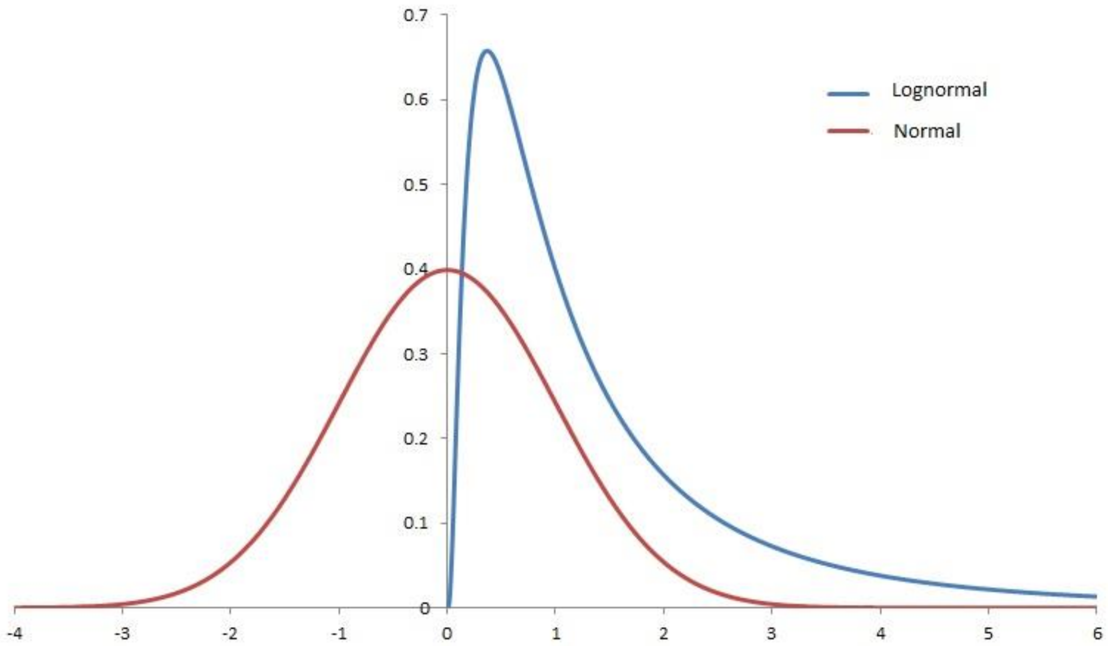
Common distributions

Normal

- Most widely used
- Symmetrical

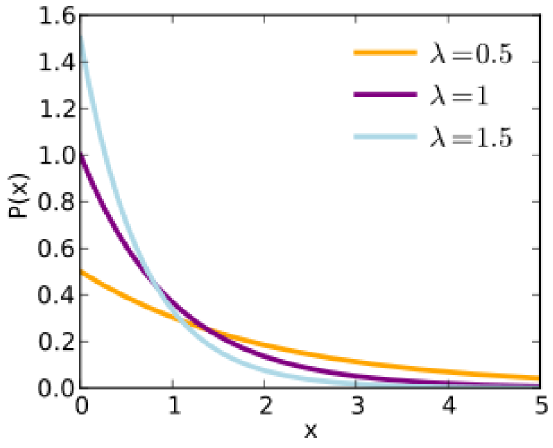
Lognormal

- Only positive values due to physical aspects of problem
- Positively skewed distribution with large spike near zero

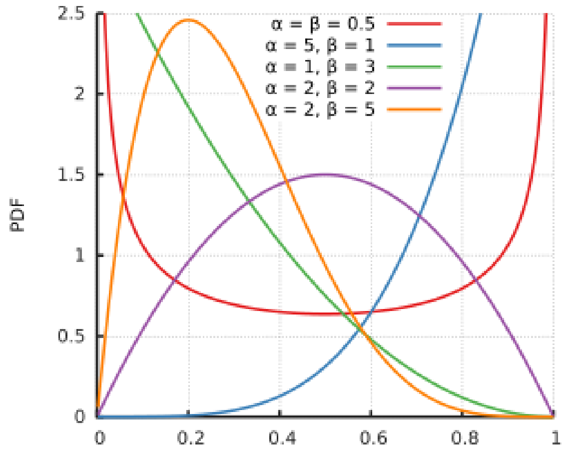


Other distributions

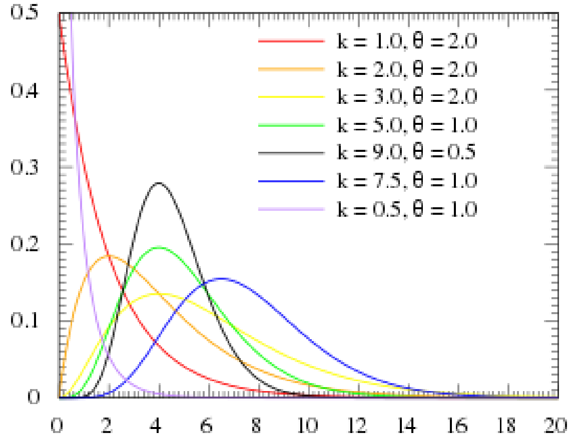
Exponential distribution



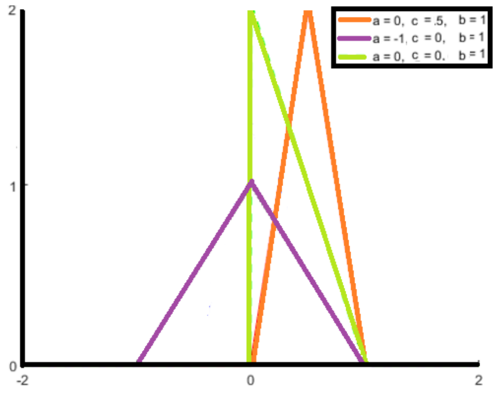
Beta distribution

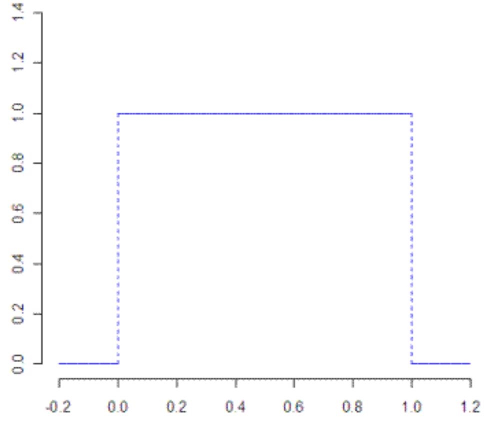


Gamma distribution



Triangular distribution

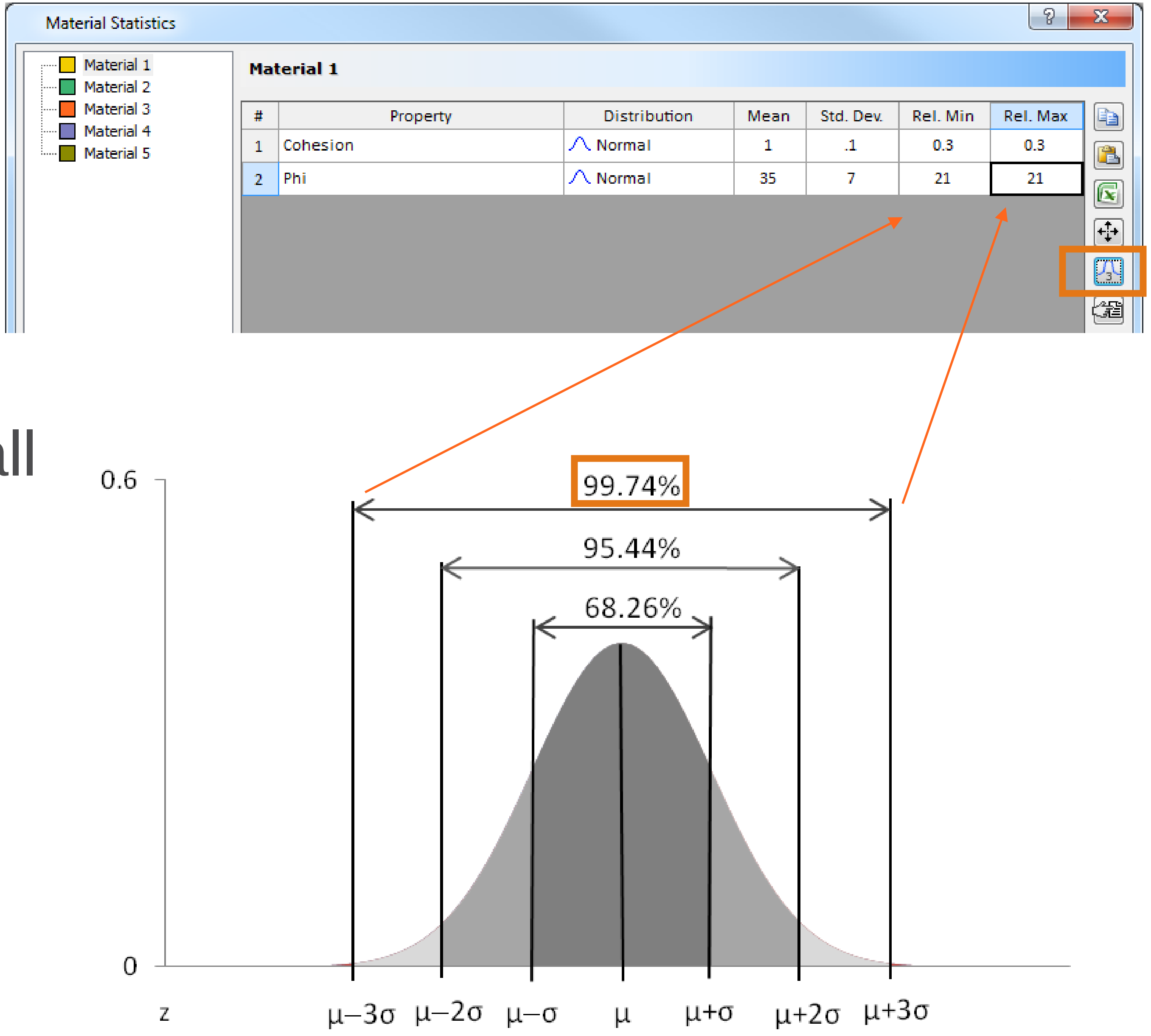




Uniform distribution

The 3σ rule

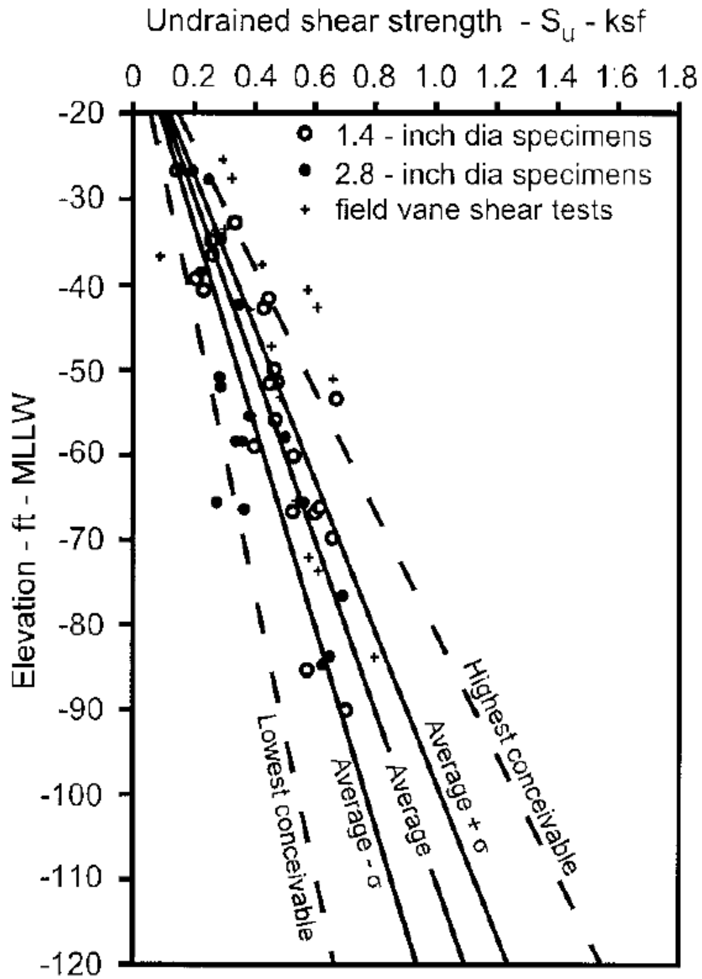
99.74% of all values of a normally distributed variable fall within plus or minus three standard deviations of the average



The 3σ rule

Can be used to estimate a value of standard deviation by first estimating the highest and lowest conceivable values (HCV, LCV) of a parameter and then dividing the difference by 6:

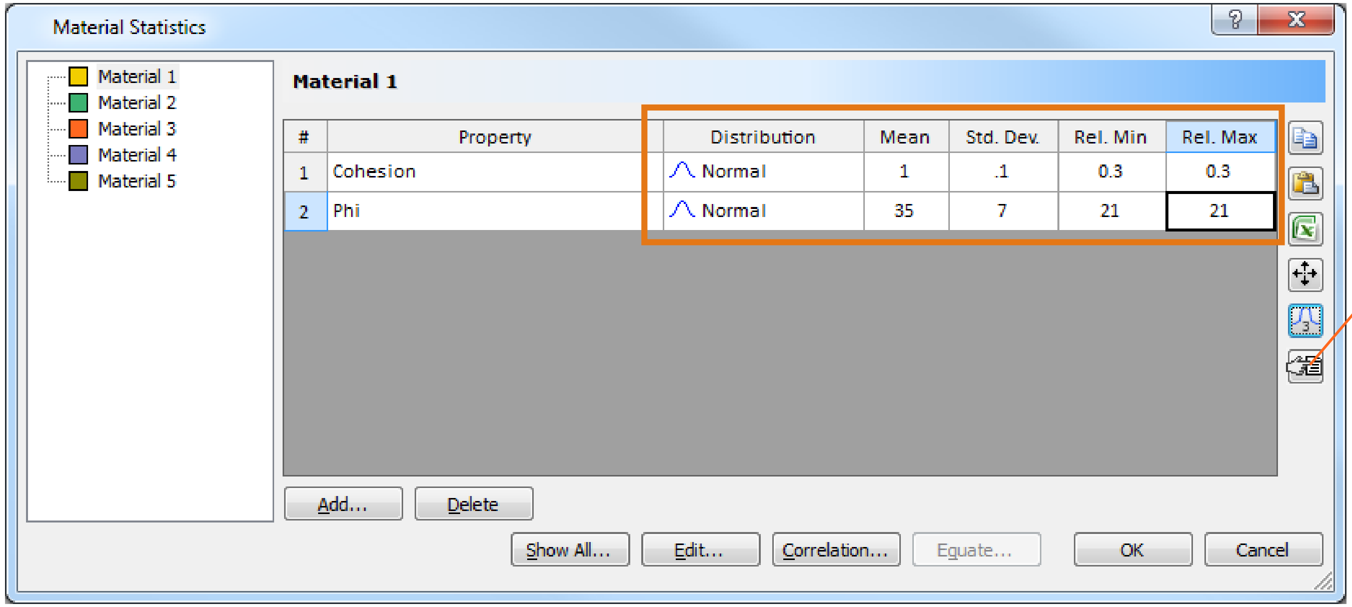
$$\sigma = \frac{HCV - LCV}{6}$$



(b) Variation in undrained shear strength with depth for San Francisco Bay mud at the LASH Terminal site in San Francisco (after Duncan and Buchignani 1973)

Probabilistic slope stability

Assign probability distributions to properties → Generate n samples for each property → Calculate FS for each simulation → Calculate PF

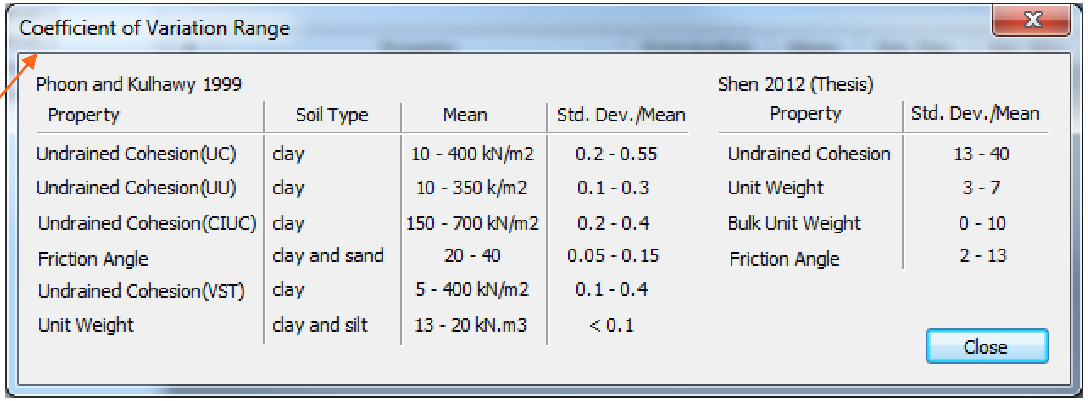


Material Statistics

- Material 1
- Material 2
- Material 3
- Material 4
- Material 5

Material 1

#	Property	Distribution	Mean	Std. Dev.	Rel. Min	Rel. Max
1	Cohesion	Normal	1	.1	0.3	0.3
2	Phi	Normal	35	7	21	21



Coefficient of Variation Range

Phoon and Kulhawy 1999

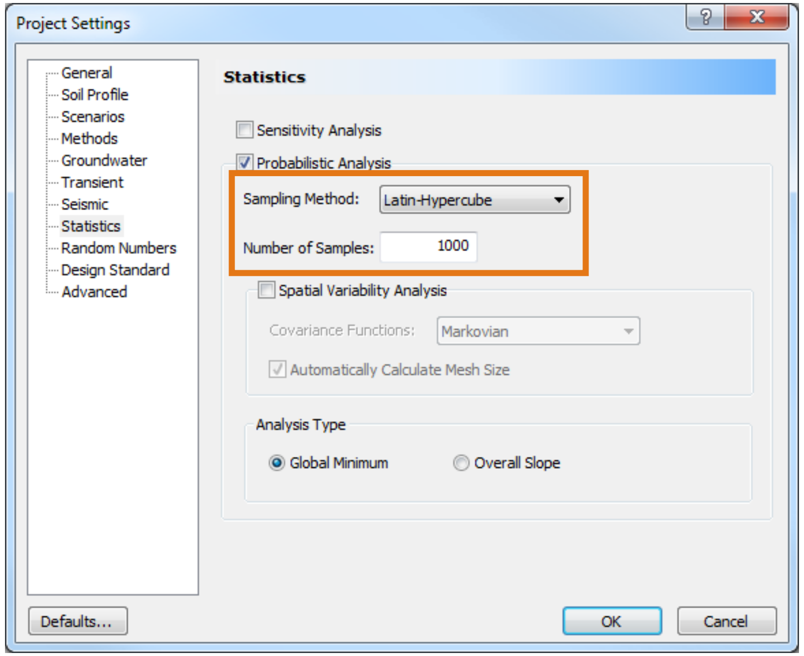
Property	Soil Type	Mean	Std. Dev./Mean
Undrained Cohesion(UC)	clay	10 - 400 kN/m2	0.2 - 0.55
Undrained Cohesion(UU)	clay	10 - 350 k/m2	0.1 - 0.3
Undrained Cohesion(CIUC)	clay	150 - 700 kN/m2	0.2 - 0.4
Friction Angle	clay and sand	20 - 40	0.05 - 0.15
Undrained Cohesion(VST)	clay	5 - 400 kN/m2	0.1 - 0.4
Unit Weight	clay and silt	13 - 20 kN.m3	< 0.1

Shen 2012 (Thesis)

Property	Std. Dev./Mean
Undrained Cohesion	13 - 40
Unit Weight	3 - 7
Bulk Unit Weight	0 - 10
Friction Angle	2 - 13

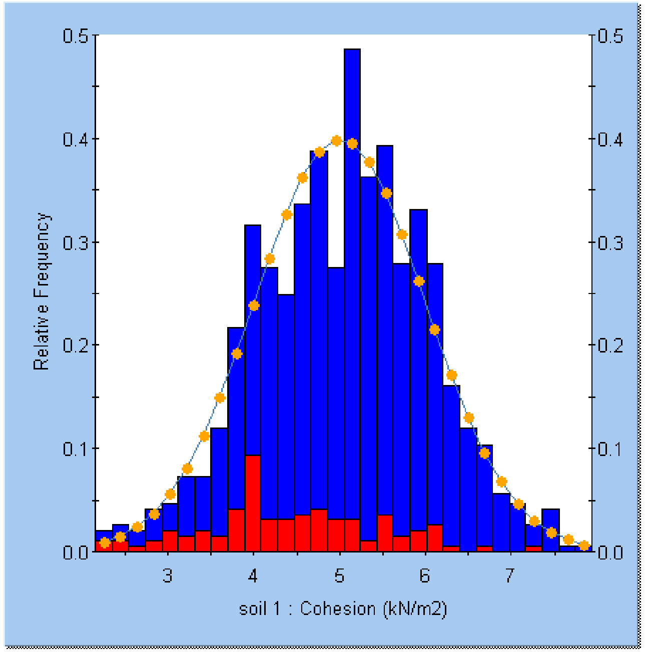
Probabilistic slope stability

Assign probability distributions to properties → Generate n samples for each property → Calculate FS for each simulation → Calculate PF



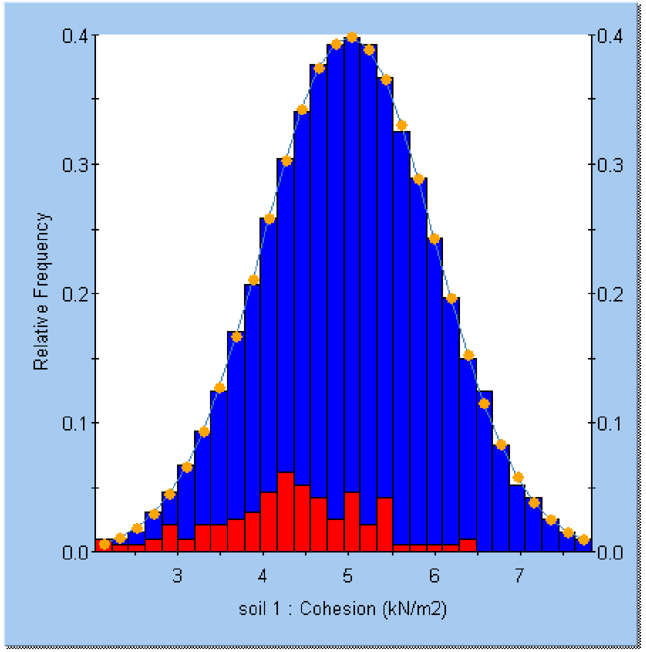
Monte Carlo vs. Latin Hypercube

Monte Carlo



SAMPLED: mean=5.035 s.d.=0.9971 min=2.154 max=7.938
INPUT: Normal mean=5 s.d.=1 min=2 max=8

Latin Hypercube



SAMPLED: mean=5 s.d.=0.9872 min=2.038 max=7.832
INPUT: Normal mean=5 s.d.=1 min=2 max=8

Requires fewer samples to give accuracy similar to Monte Carlo method

Number of samples

1) Using Formulas

$$n = \left[\frac{(d^2)}{4(1-\varepsilon)^2}\right]^m$$

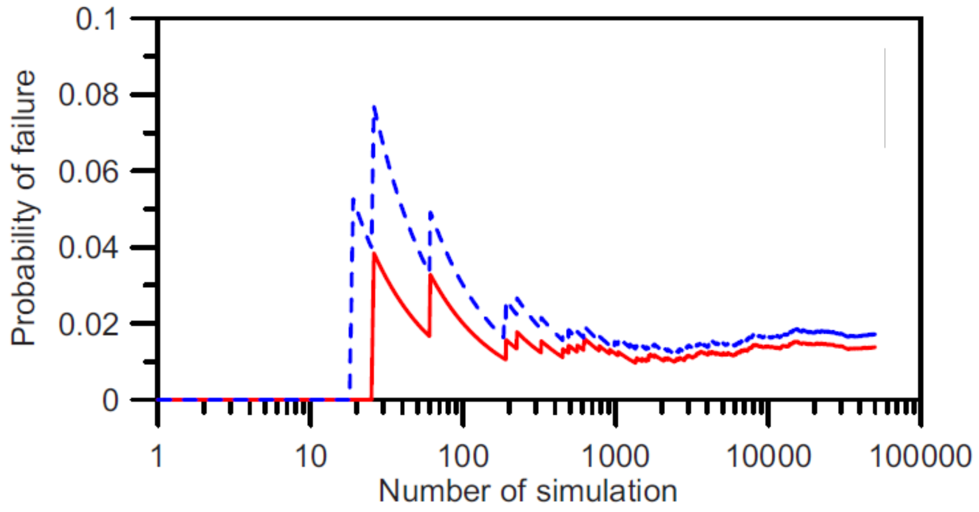
n = number of Monte Carlo samples

m = number of random variables

d = normal standard deviate

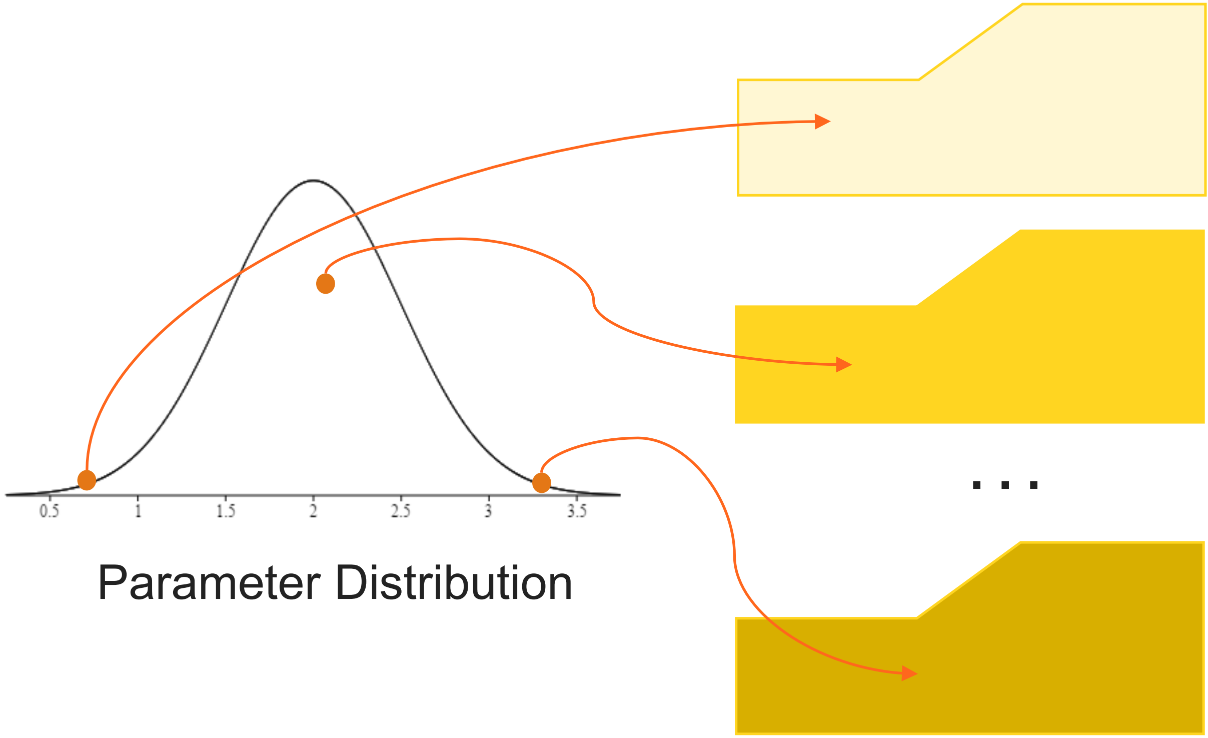
ε = level of confidence

2) Sensitivity analysis



Probabilistic slope stability

Assign probability distributions to properties → Generate n samples for each property → Calculate FS for each simulation → Calculate PF

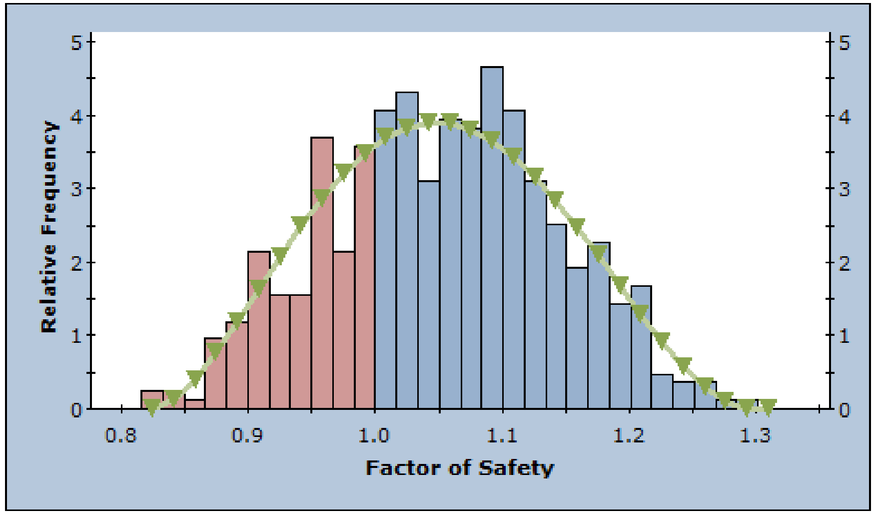


Probabilistic slope stability

Assign probability distributions to properties → Generate n samples for each property → Calculate FS for each simulation → Calculate PF

$$PF = \frac{\#\ of\ simulations\ with\ FS < 1}{Total\ \#\ of\ simulations}$$

Highlighted Data = Factor of Safety < 1 (146 of 500 samples)



Probability of Failure: 0.292
BEST FIT: mean = 1.05226 fit = Beta s.d. = 0.0905499 min = 0.824479 max = 1.29539

Comparison to literature

$\phi = 30^\circ$

Slope angle (α) = 60°

$\mu_c/(\mu_\gamma H \tan\mu_\phi) = 0.2$

COVc = 0.5

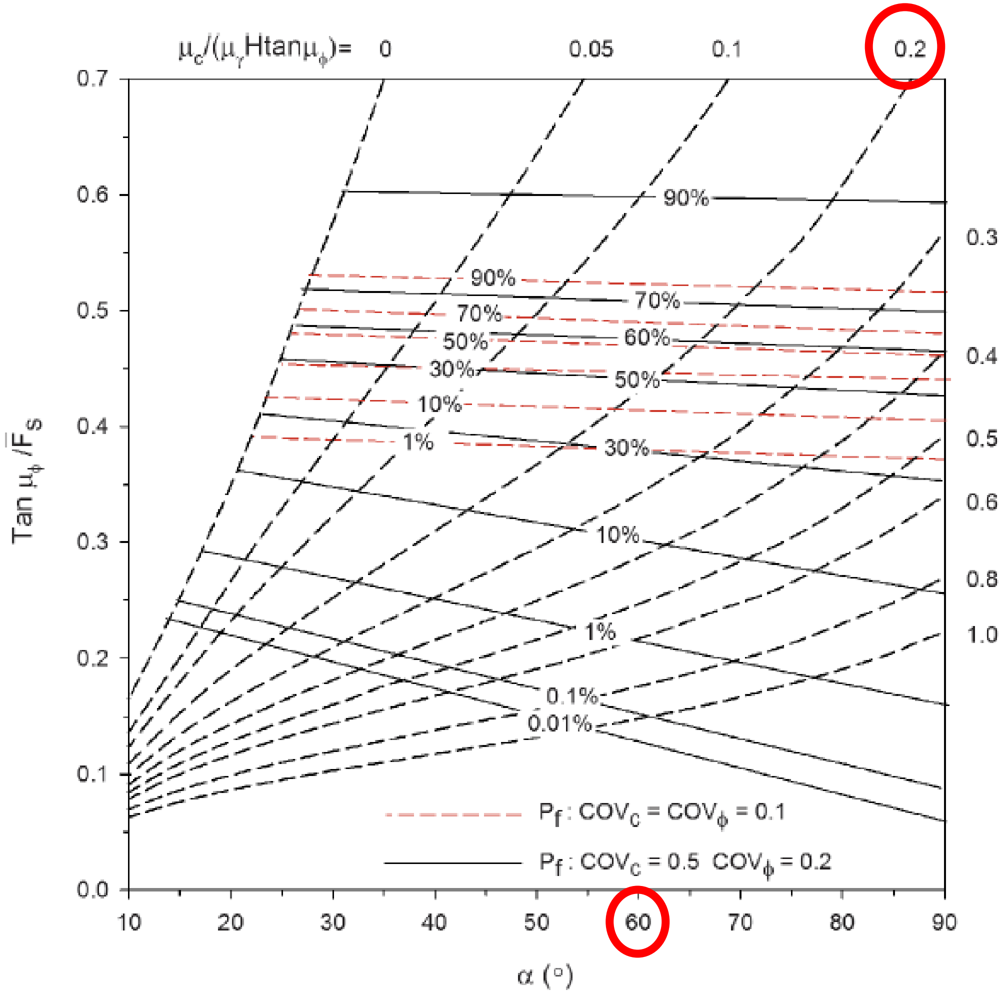
COVϕ = 0.2

FS = 1.3

PF ≈ 25% (using chart)

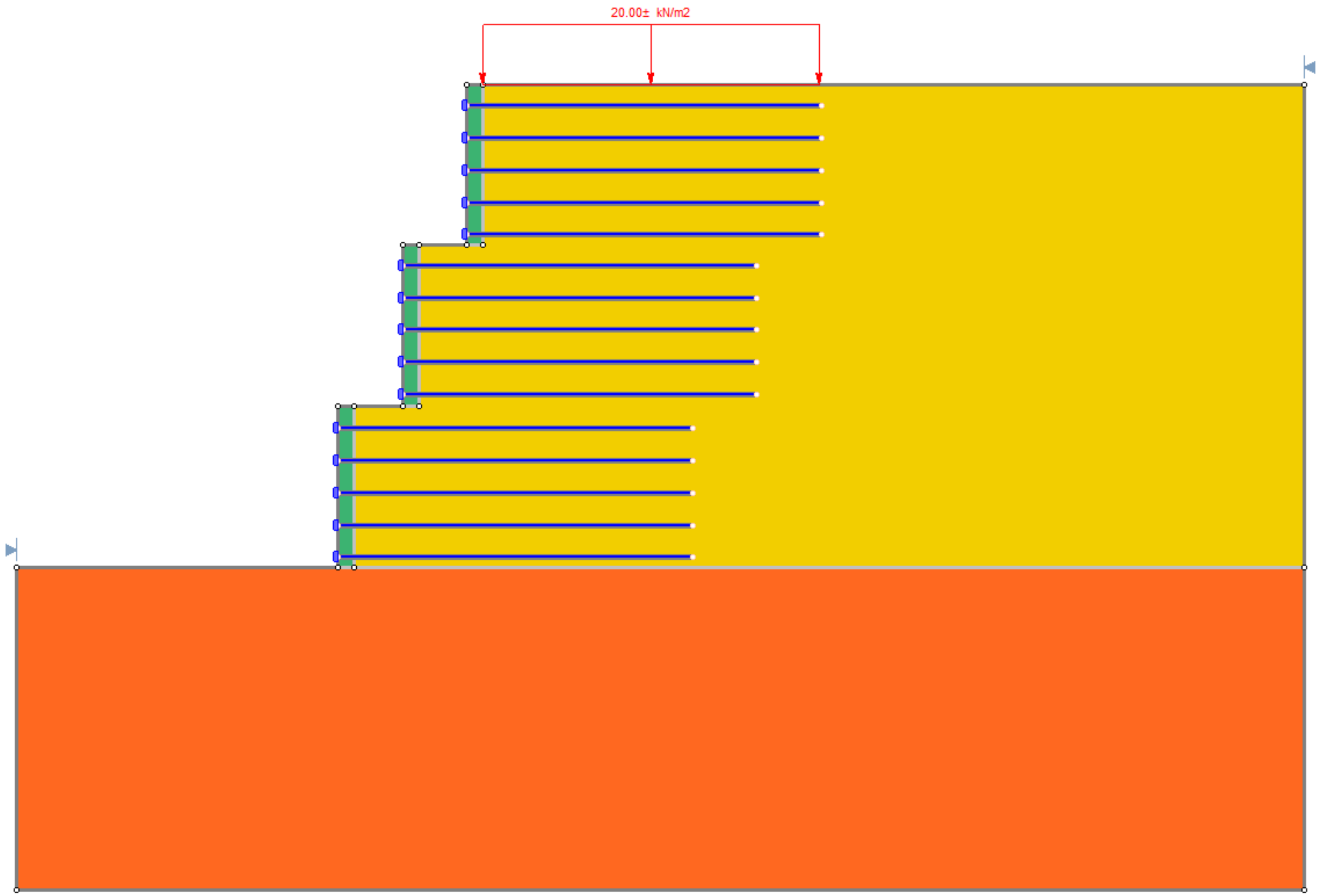
FS = 1.33

PF = 23.45 % (using *Slide2*)



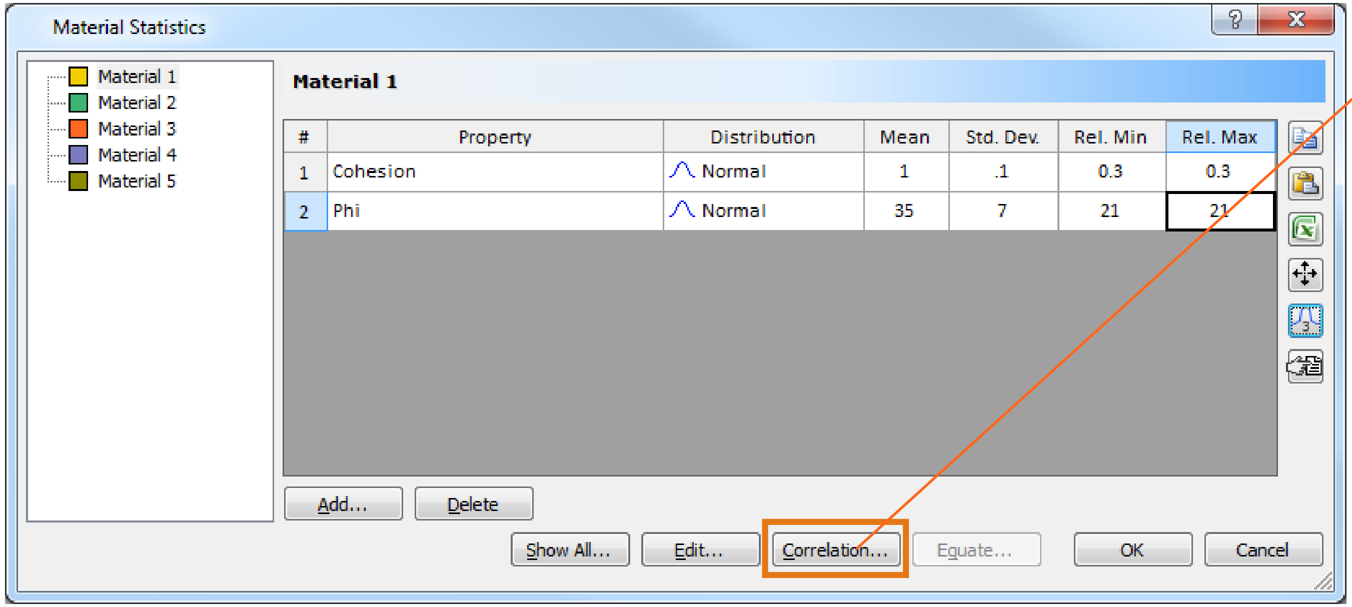
Simple Examples with *Slide2*

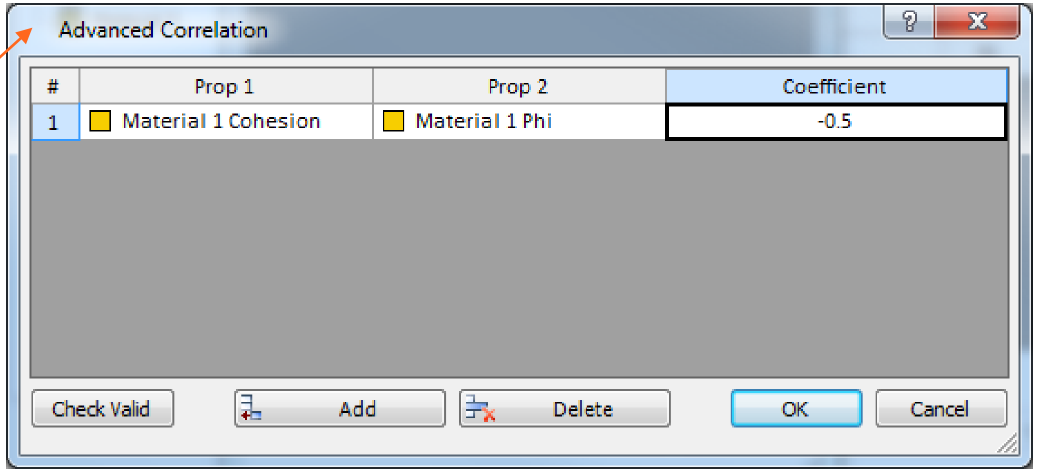
Example 1: Reinforced retaining wall

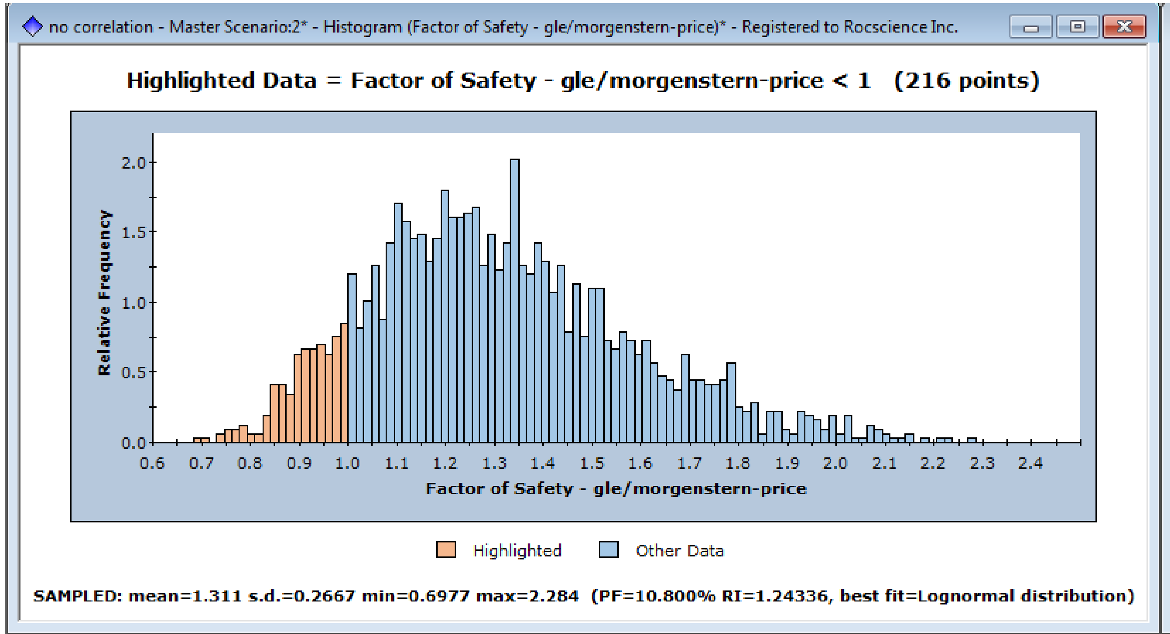


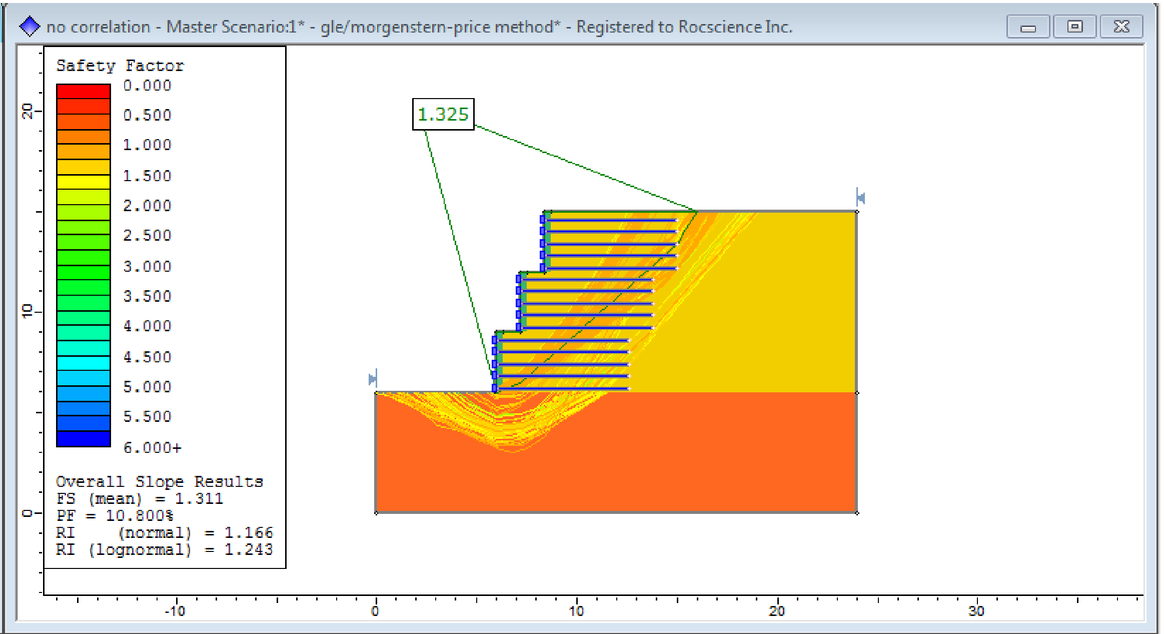
Probabilistic slope stability

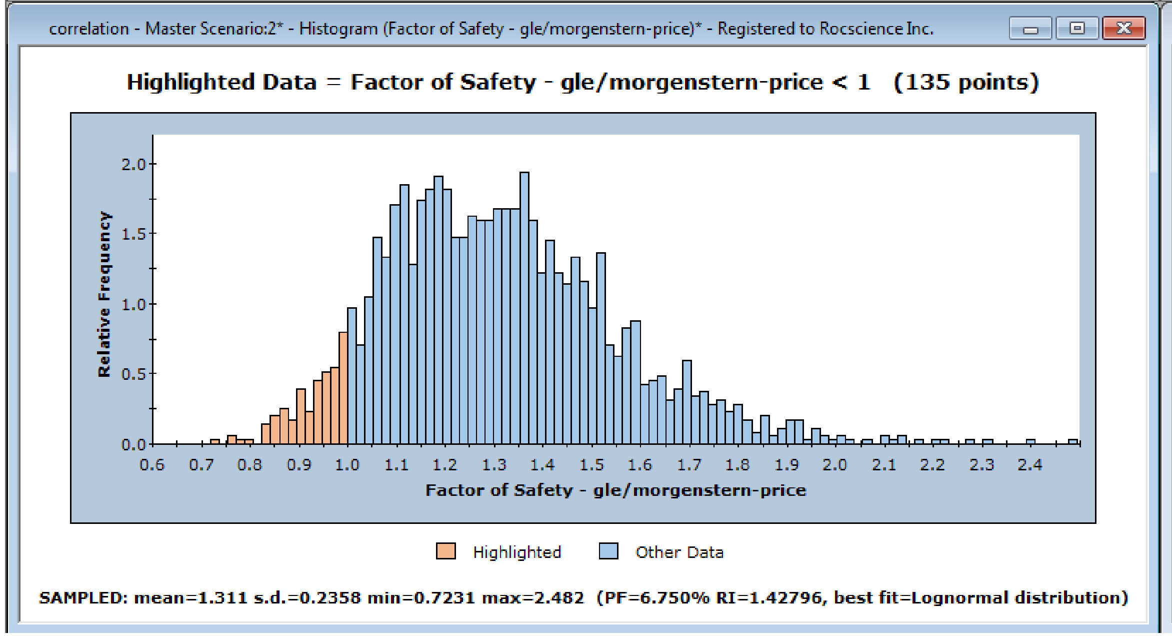
Assign probability distributions to properties → Define correlation → Generate n samples for each property → Calculate FS for each simulation → Calculate PF

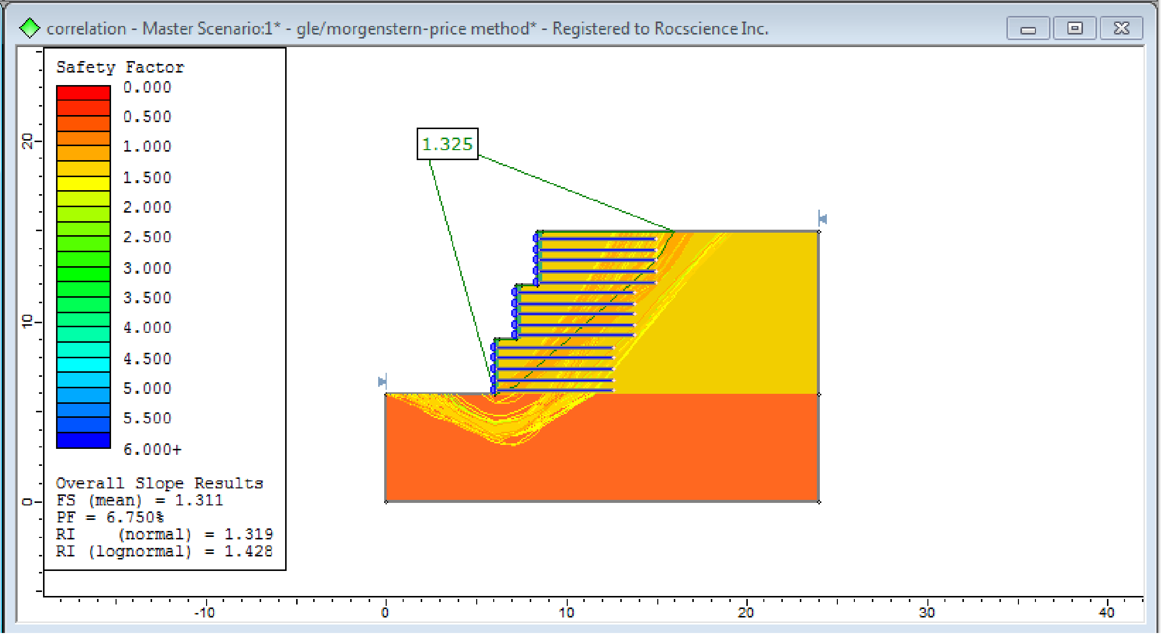





no correlation - Master Scenario:2* - Histogram (Factor of Safety - gle/morgenstern-price)* - Registered to Rocscience Inc.
Highlighted Data = Factor of Safety - gle/morgenstern-price < 1 (216 points)
Relative Frequency
Factor of Safety - gle/morgenstern-price
Highlighted
Other Data
SAMPLED: mean=1.311 s.d.=0.2667 min=0.6977 max=2.284 (PF=10.800% RI=1.24336, best fit=Lognormal distribution)


no correlation - Master Scenario:1* - gle/morgenstern-price method* - Registered to Rocscience Inc.
Safety Factor
Overall Slope Results
FS (mean) = 1.311
PF = 10.800%
RI (normal) = 1.166
RI (lognormal) = 1.243
1.325


correlation - Master Scenario:2* - Histogram (Factor of Safety - gle/morgenstern-price)* - Registered to Rocscience Inc.
Highlighted Data = Factor of Safety - gle/morgenstern-price < 1 (135 points)
Relative Frequency
Factor of Safety - gle/morgenstern-price
Highlighted
Other Data
SAMPLED: mean=1.311 s.d.=0.2358 min=0.7231 max=2.482 (PF=6.750% RI=1.42796, best fit=Lognormal distribution)


correlation - Master Scenario:1* - gle/morgenstern-price method* - Registered to Rocscience Inc.
Safety Factor
Overall Slope Results
FS (mean) = 1.311
PF = 6.750%
RI (normal) = 1.319
RI (lognormal) = 1.428
1.325

Probabilistic in Tunneling

Section 3

Probabilistic Analysis in RS2 (Tunneling example)

Overview

- In *RS2* you can perform a **probabilistic analysis** to determine the effect of uncertainty or variability of input parameters, on the results of the finite element analysis.
- Three sampling methods available:
  - Point Estimate
  - Monte Carlo
  - Latin Hypercube
- The following model properties can be defined as random variables
  - Material Properties
  - Joint Properties
  - Field Stress

Point Estimate Method

- The **Point Estimate Method** is a simplified method of generating random input variables based on Rosenblueth’s point estimate method.
- In this method, two "point estimates" are made for each random variable at fixed values of **one standard deviation** on either side of the mean (mean + standard deviation, mean - standard deviation).
- The finite element analysis is carried out for each possible combination of point estimates. This produces 2^m solutions, where **m** is the number of random variables involved.

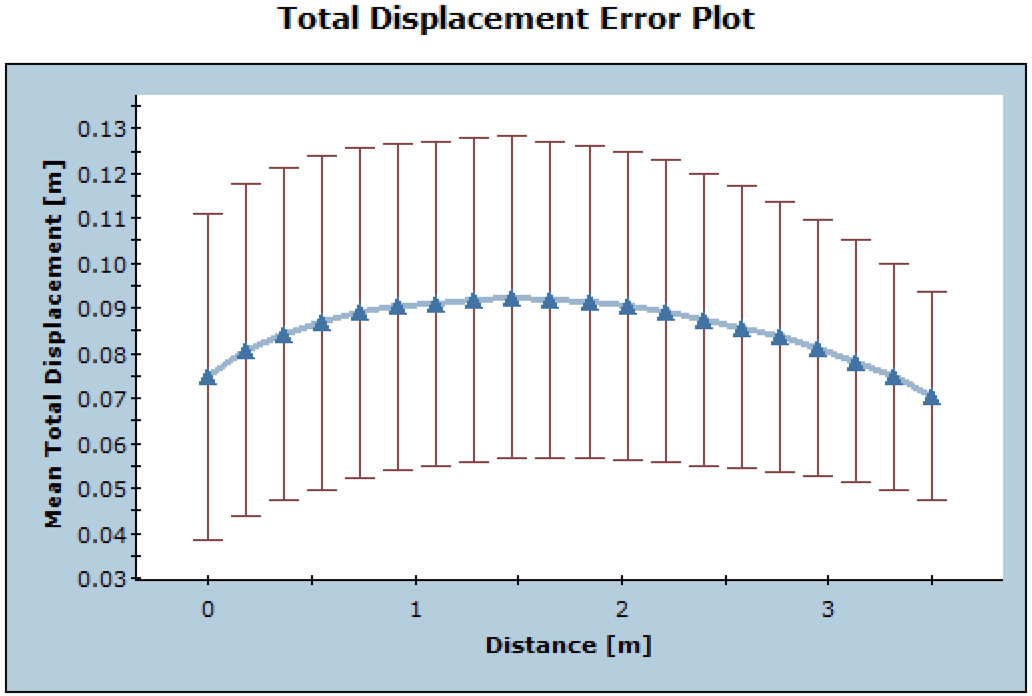
Point Estimate Method

- In the context of geotechnical finite element analysis, the Rosenblueth point estimate method of probabilistic analysis is suitable for problems involving only a few random variables (e.g. 2 to 6).
- For larger numbers of random variables (e.g. greater than 10) the computation time and output file storage requirements may become prohibitive. This will depend on the size and nature of the problem being computed.
- For problems involving more than about 10 random variables, the **Monte Carlo** or **Latin Hypercube** methods are recommended.

Error Plot

- The **Error Plot** option allows you to plot the mean and standard deviation of data along any material query after a Probabilistic Analysis.
- The central curve plots the mean data values along the query, and the height of the vertical bars indicate plus / minus one standard deviation from the mean values.

Error plot for total displacement along floor of tunnel

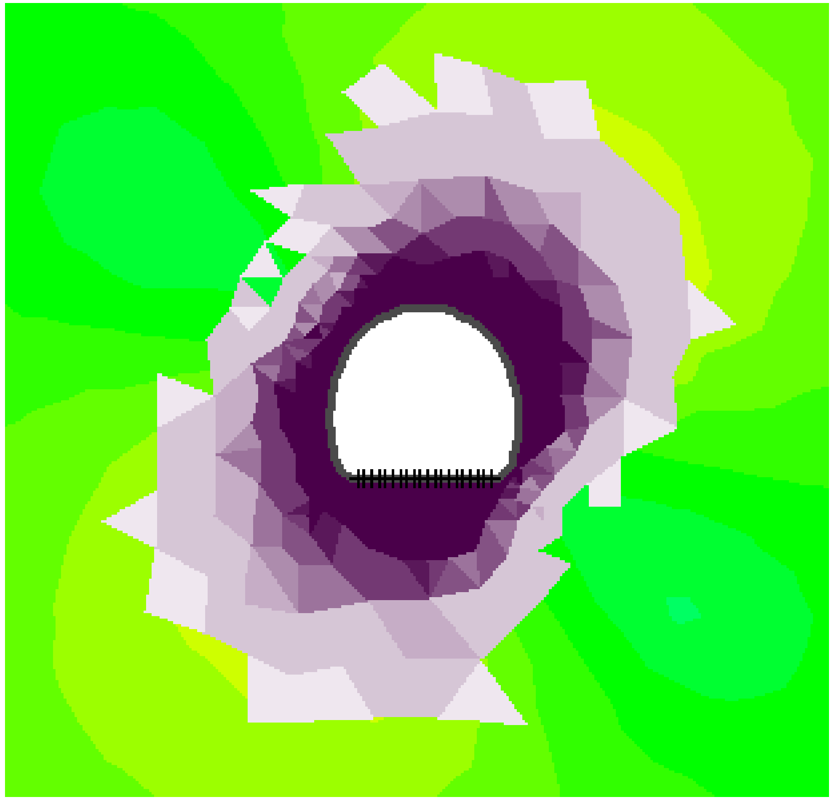


Show Yield Zones

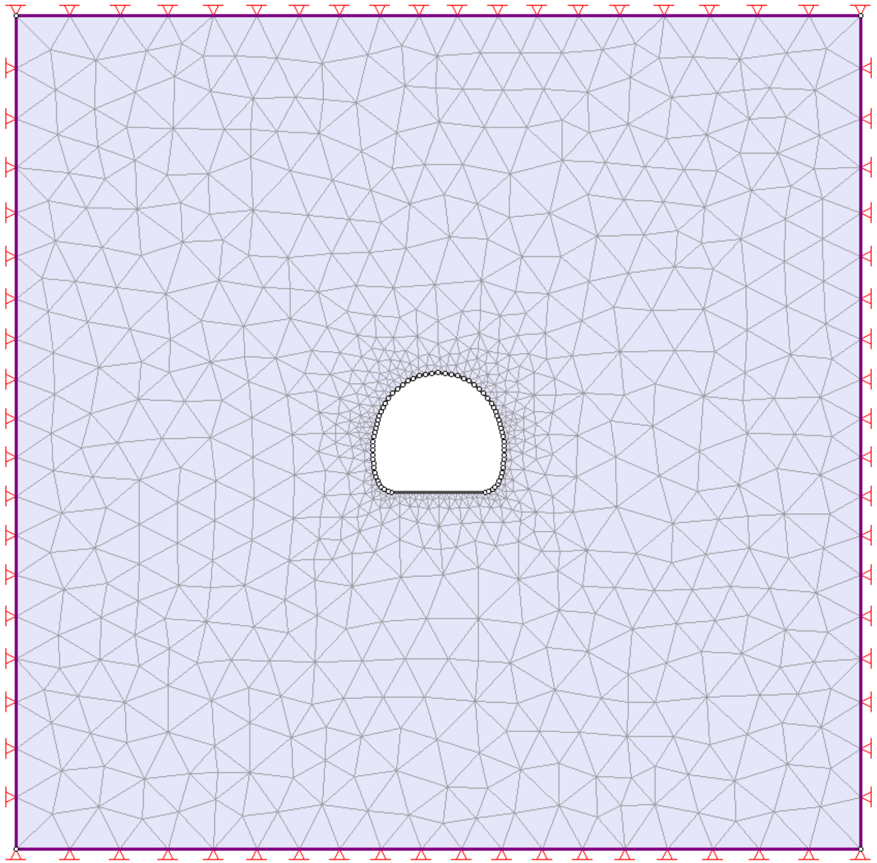
The **Show Yield Zones** option will highlight all yielded finite elements after a probabilistic analysis.

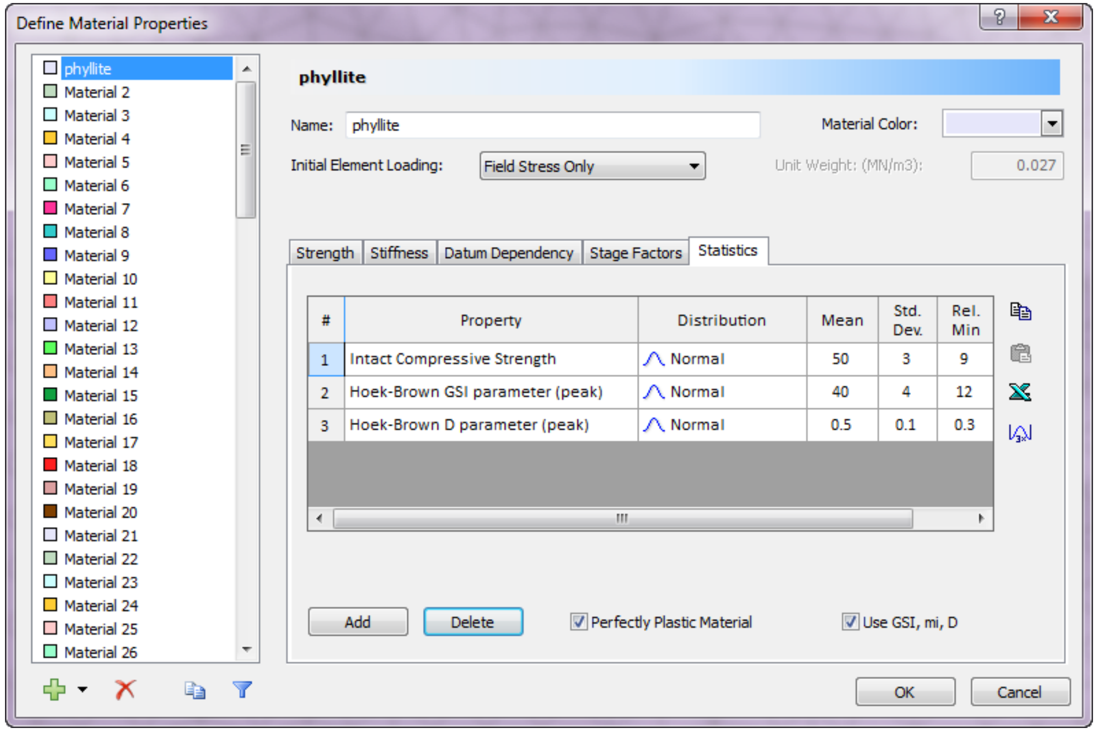
- All yielded finite elements from all component files of the probabilistic analysis will be highlighted (i.e. if an element failed during any run of the probabilistic analysis it will be highlighted).
- The shading of the elements indicates the probability of failure of the elements (darker shading indicates higher

Display of yielded elements with Show Yield Zones



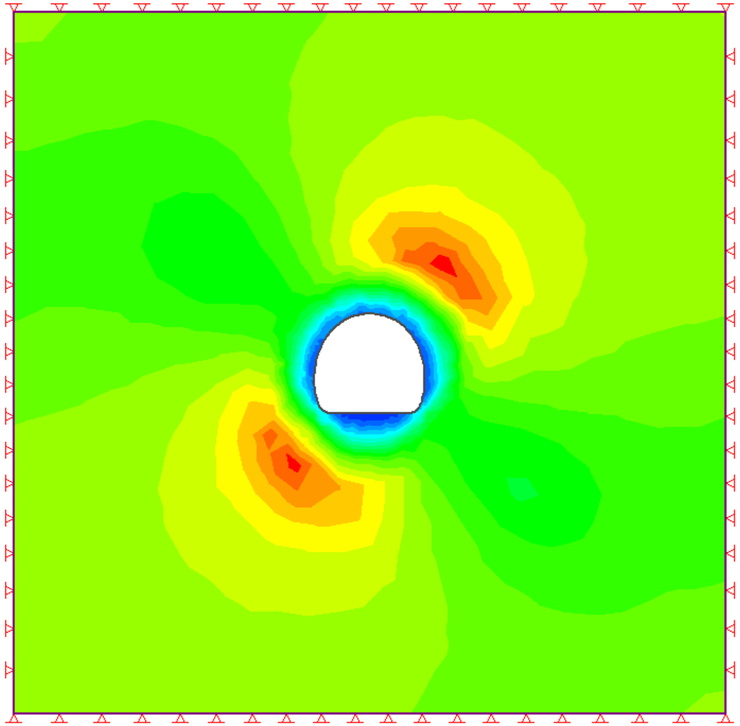
Underground Tunnel



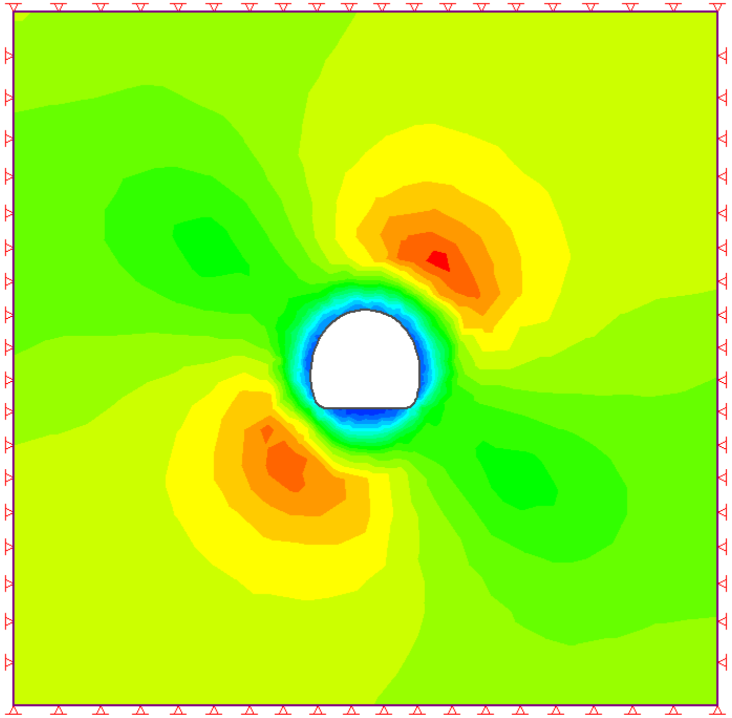


Statistical Results – Sigma 1

Base File Values

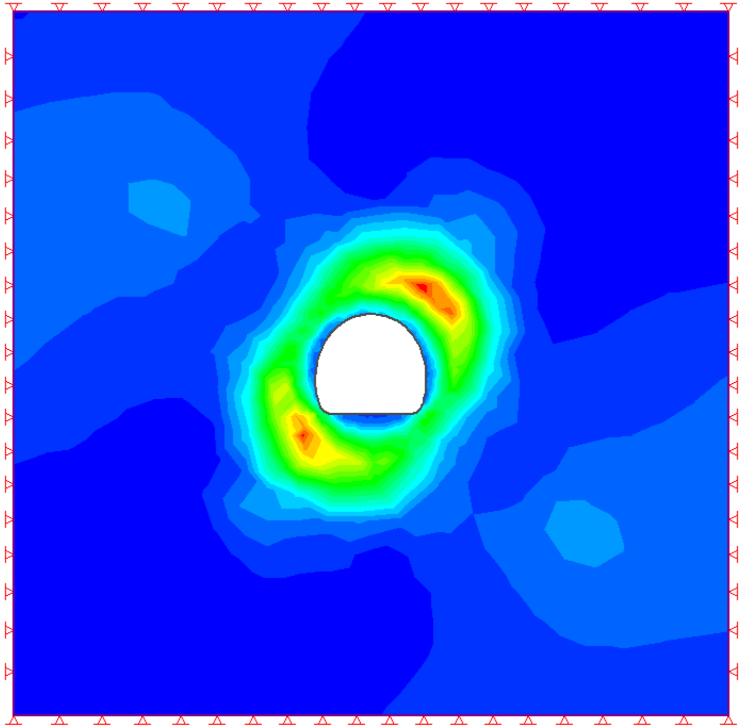


Mean Values

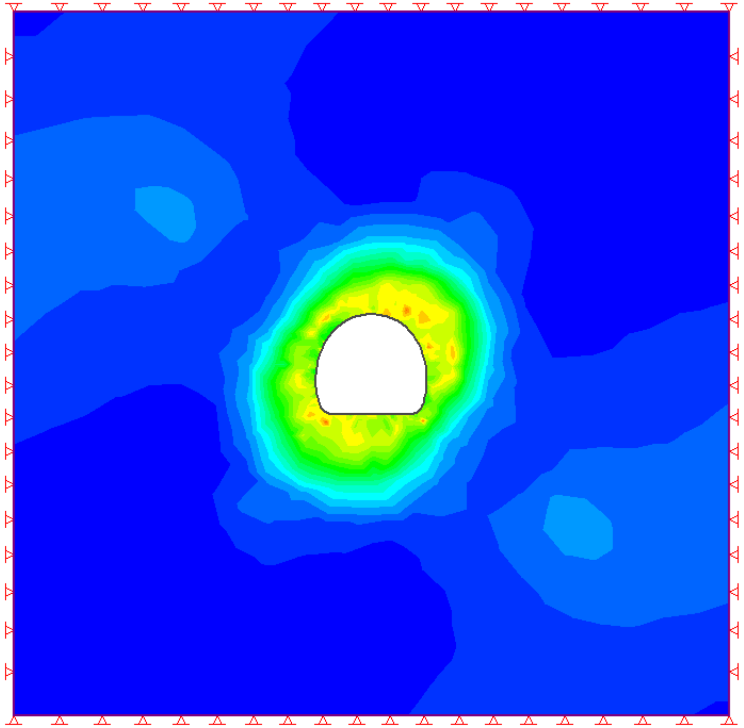


Statistical Results – Sigma 1

Standard Deviation Values



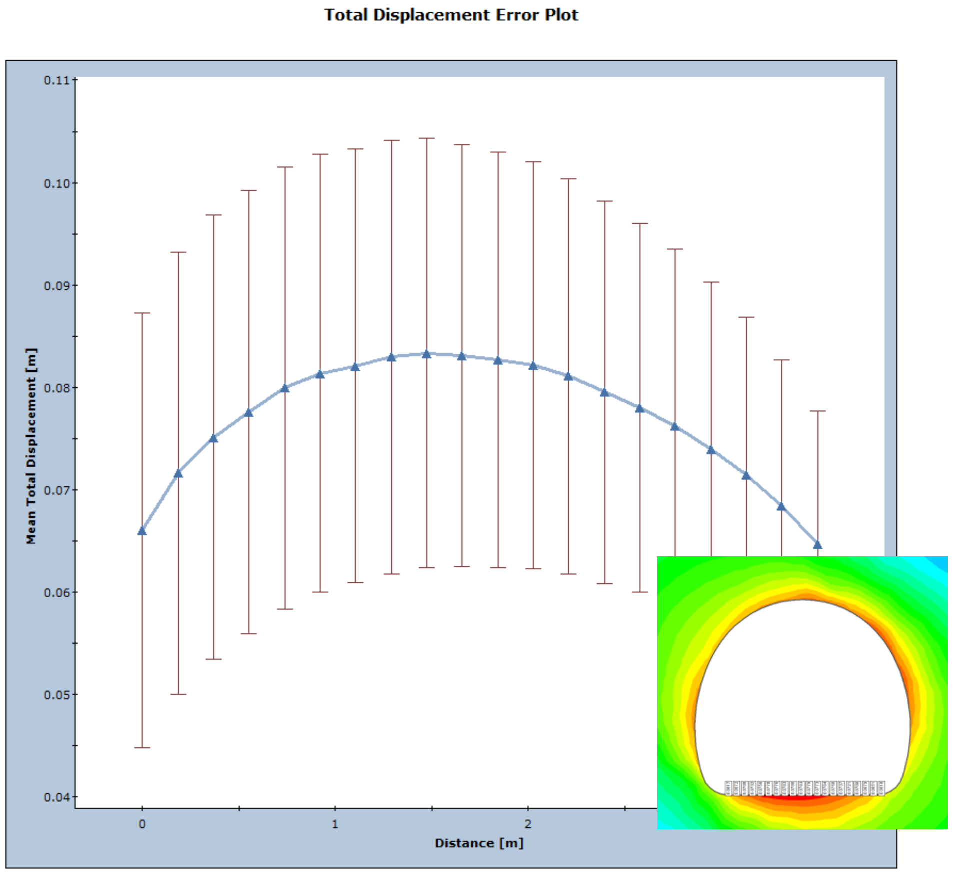
Coefficient of Variation



Show Model Yield Zones

- The darkest color indicates elements that fail in every model.
- Lighter colors indicate elements that are less likely to fail.
- Plot is useful when trying to determine the extent of possible failure when designing rock bolt support, for example.
- The shading of the yield zones can be customized in the Show Yield Options dialog in the Statistics menu.

Total Displacement Error Plot



- Plot shows mean displacement along the bottom of the tunnel with error bars indicating on standard deviation of displacement.
- The error plot indicates the range of possible floor displacements that can be expected for the given uncertainty in material properties.

Spatial variability analysis

Section 3

Spatial variability of soil strength parameters

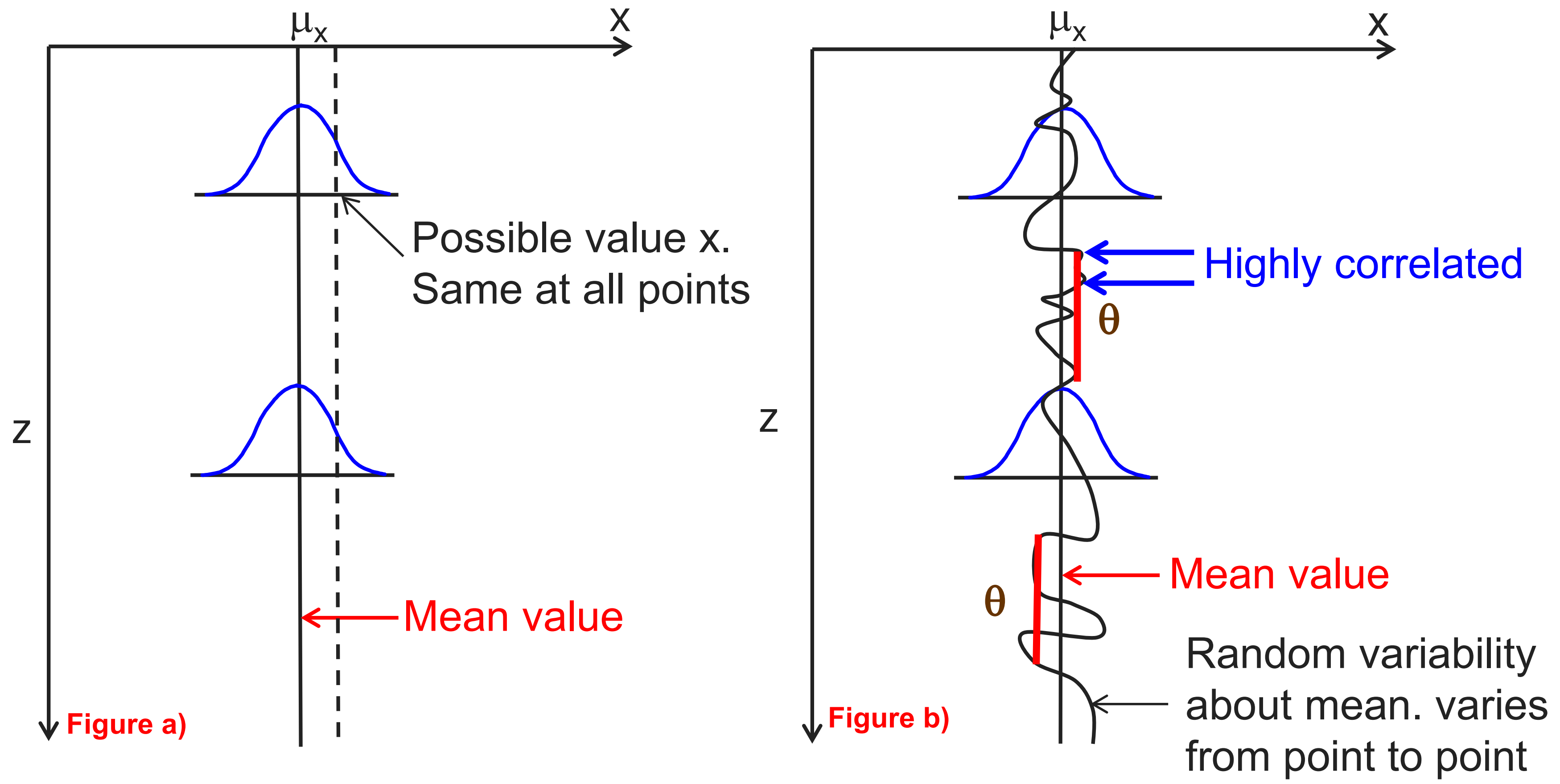


Figure a) **The soil property is perfectly correlated throughout the soil profile (statically uniform)**

Figure b) **Two samples taken from close locations have highly correlated properties while as the**

Spatial correlation length (θ):

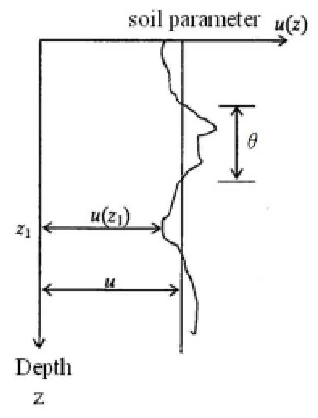
Spatial variability of soil properties

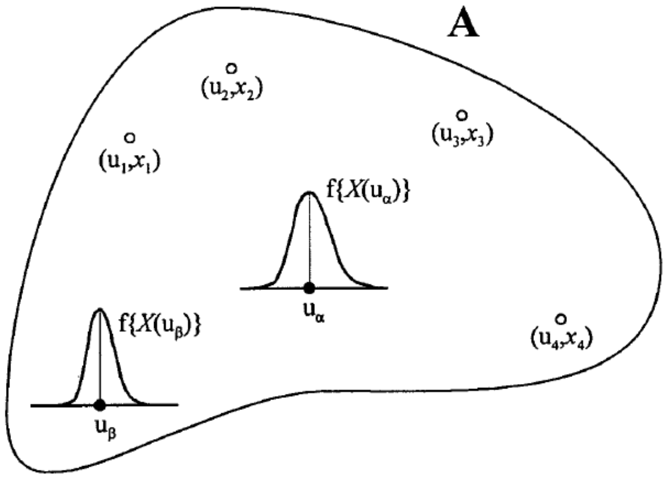
Parameter X is defined by location vector and the magnitude of the parameter

X at any location ua is a random variable

Fenton and Griffiths (2008):

1) 2D exponentially decaying (Markov) model
2) 2D separable (1Dx1D) Markov model
3) 2D separable Gaussian decaying model
4) 2D isotropic fractional Gaussian noise model
5) 2D separable fractional Gaussian noise model

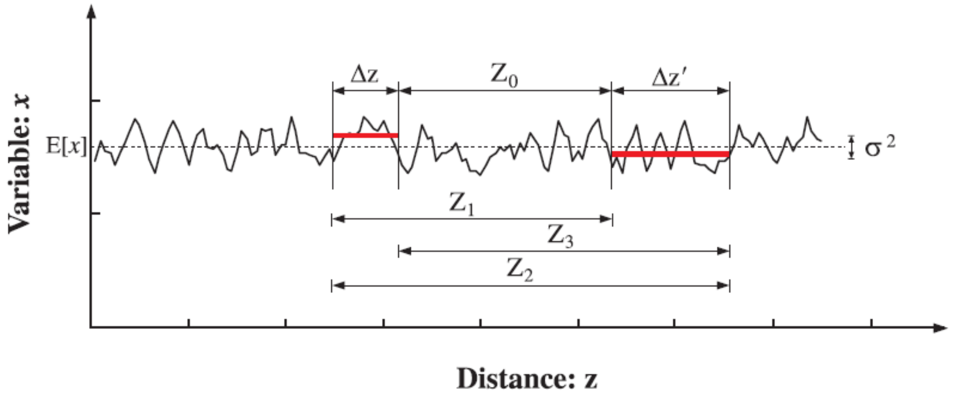




Random field theory

Local average subdivision method:

Limiting equilibrium failures of slopes depend upon the average strength across the failure surface.



$$\rho(X_{\Delta z}, X'_{\Delta z'}) = \frac{Z_o^2\Gamma(Z_o) - Z_1^2\Gamma(Z_1) + Z_2^2\Gamma(Z_2) - Z_3^2\Gamma(Z_3)}{2\Delta z\Delta z'\,[\Gamma(\Delta z)\Gamma(\Delta z')]^{0.5}}$$

El-Ramly (2001)

Variance reduction method

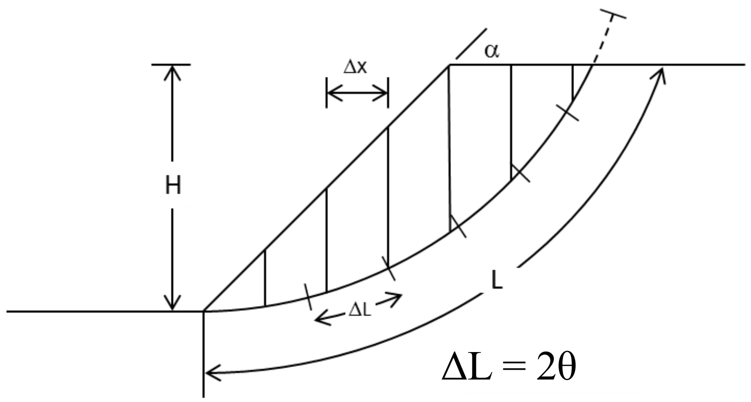
The variance reduction function is a measure of the reduction in the point variance under spatial average.

$$\Gamma(\Delta z) = \begin{cases} 1 & \Delta z \leq \delta \\ \dfrac{\delta}{\Delta z} & \Delta z \geq \delta \end{cases}$$

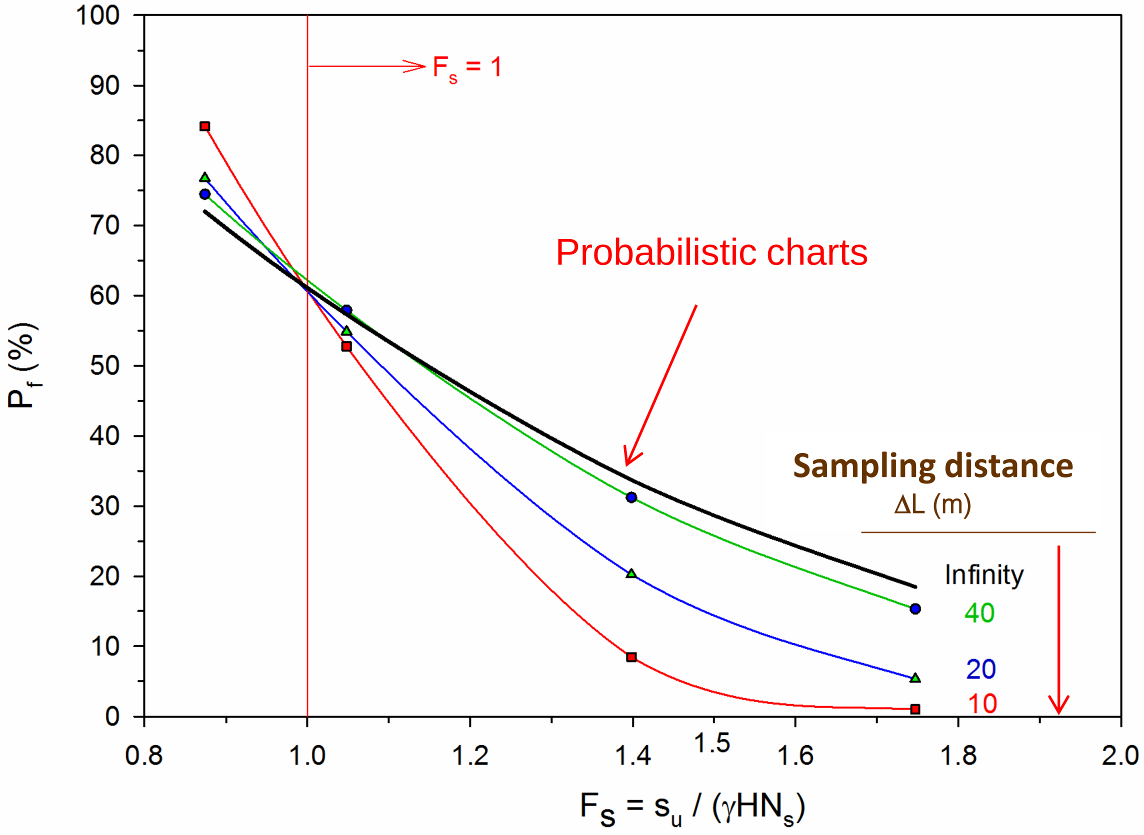
Spatial variability of soil strength parameters

Effect of spatial variability of soil properties on Pf

Probabilistic charts: Spatial correlation length of infinity



1D spatial variability



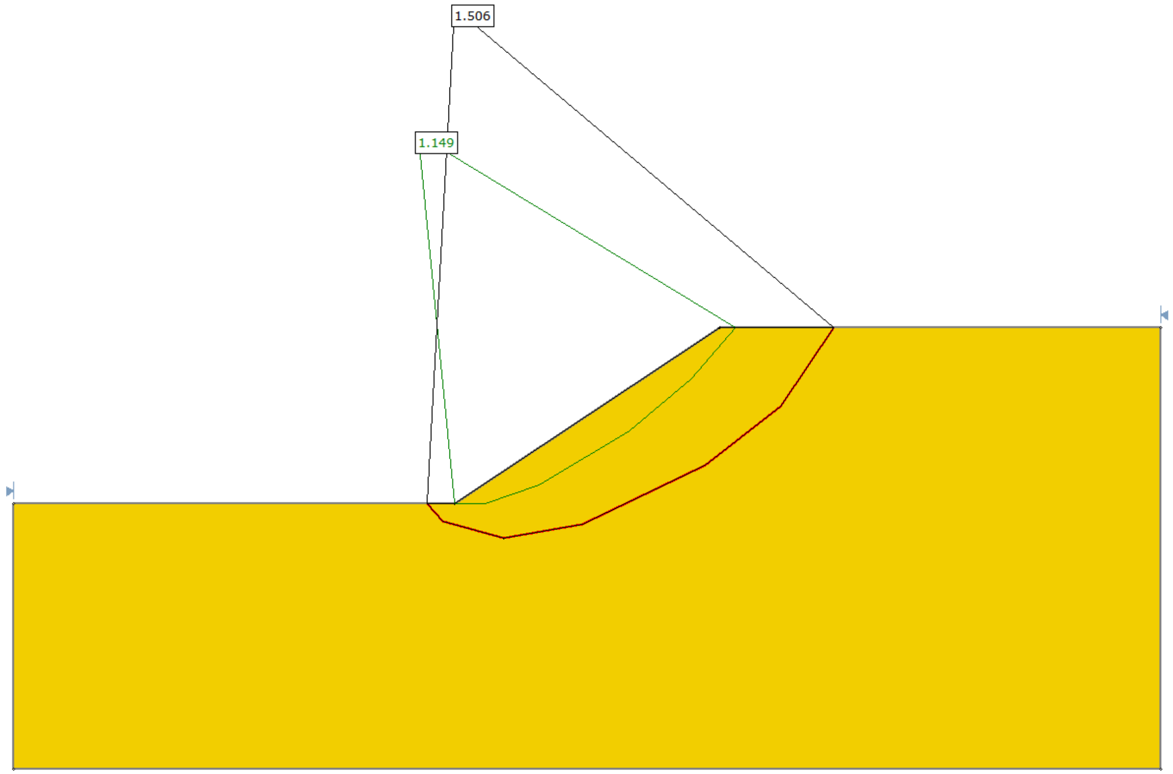
Surface Altering Optimization technique (SAO)

General Overview

- Surface Optimization: is a powerful tool to yield lower factors of safety by optimizing geometry of a given non-circular surface
  - ✓ surface can be: 1) output of a non-circular search method
    2) user-define
- Monte Carlo random-walk has been the mostly used optimization method in practice (Greco 1996).
- Surface Altering Optimization is a new alternative approach which is based on BOBYQA, a derivative-free nonlinear optimization method developed by Powell (2009).

Problem Statement

- Given a set of points describing surface, we are interested in moving surface points to achieve the minimum FS

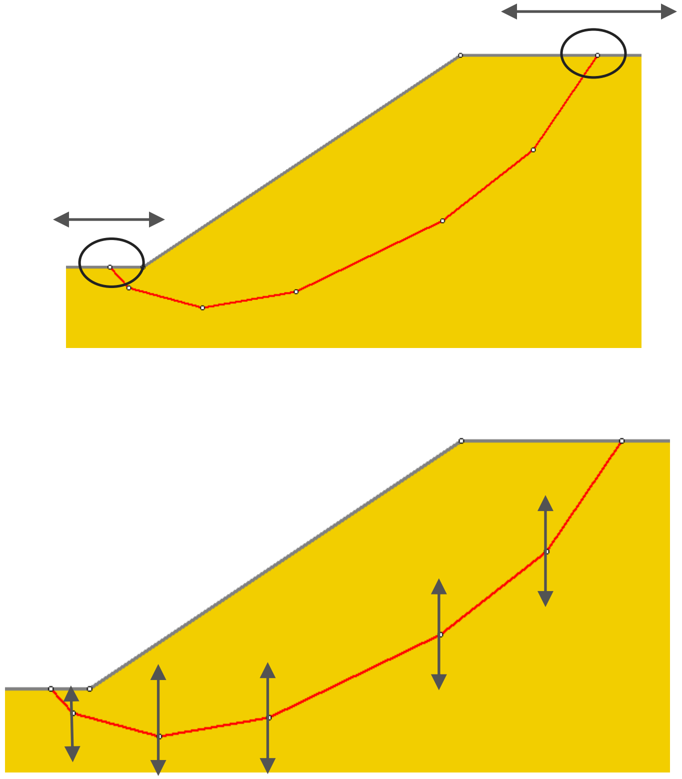


Surface Altering Optimization

There are two iterative steps to alter surface with the goal of minimizing FS:

1) Relocating end points

2) Adjusting y-ordinates of points between the two ends



Surface Altering Optimization

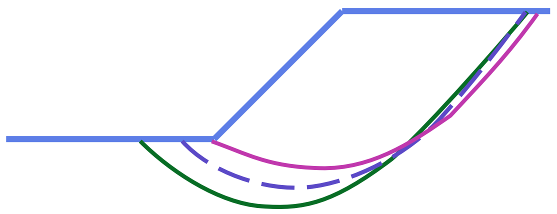
- Each step, uses **BOBYQA** to minimize FS

- **BOBYQA** is a **constrained derivative-free** non-linear optimization technique based on **trust-region** approach

- In general, this approach requires significantly smaller number of function evaluations compared to random walk

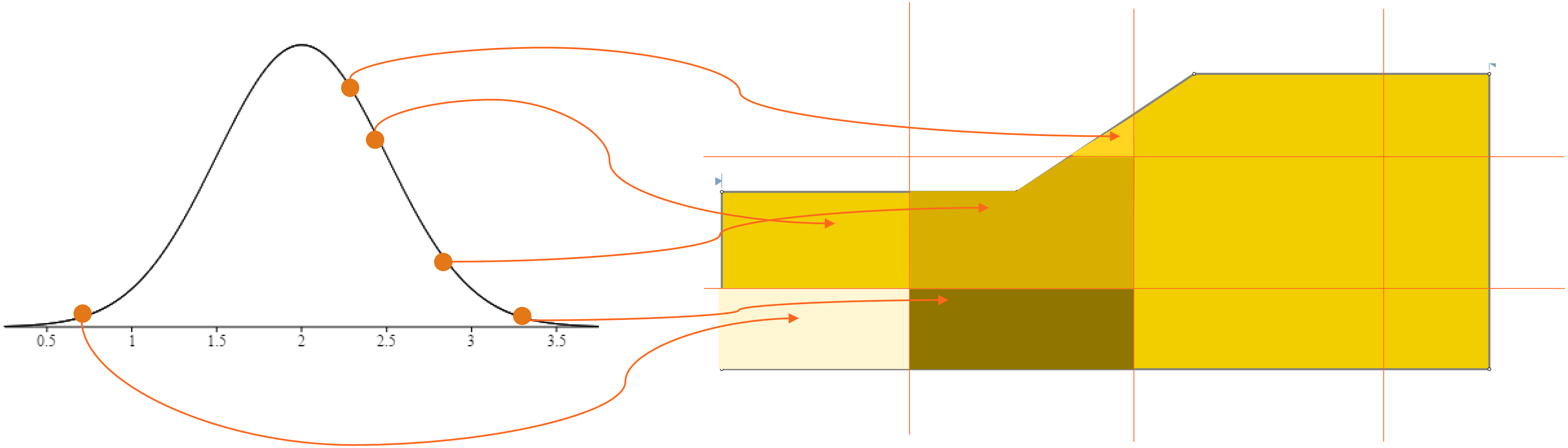
Surface Altering Optimization (SAO)

- SAO can be combined with all the search methods
- SAO is a powerful tool to yield lower factors of safety by modifying geometry of a given surface
- SAO is based on a sequence of transformations applied to the geometry of the input surface as a whole



2D Spatial variability

Can we be even more realistic?

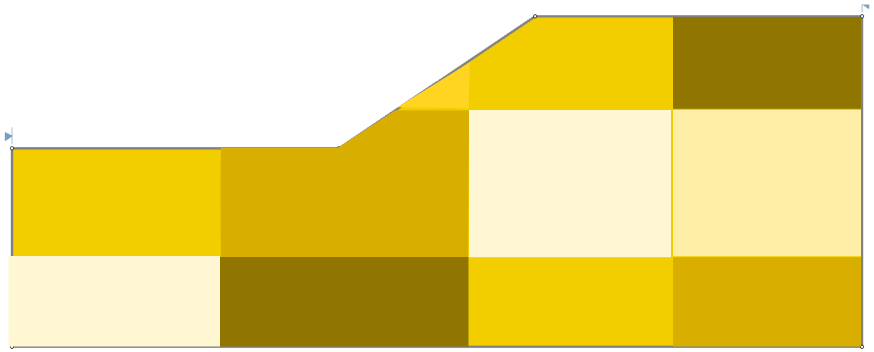


Comparison of samples

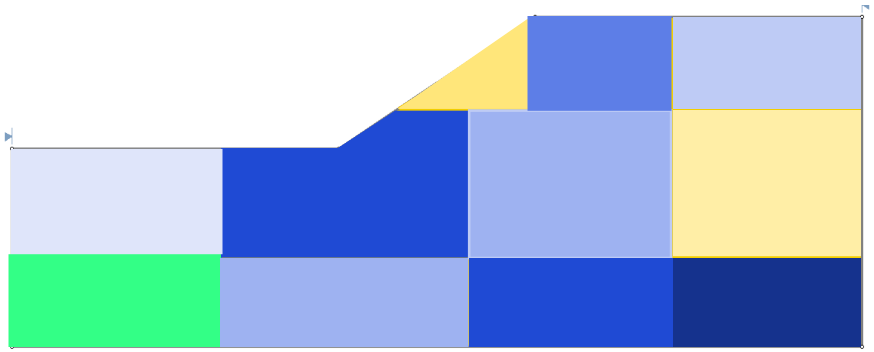
SRV

Spatial Variability

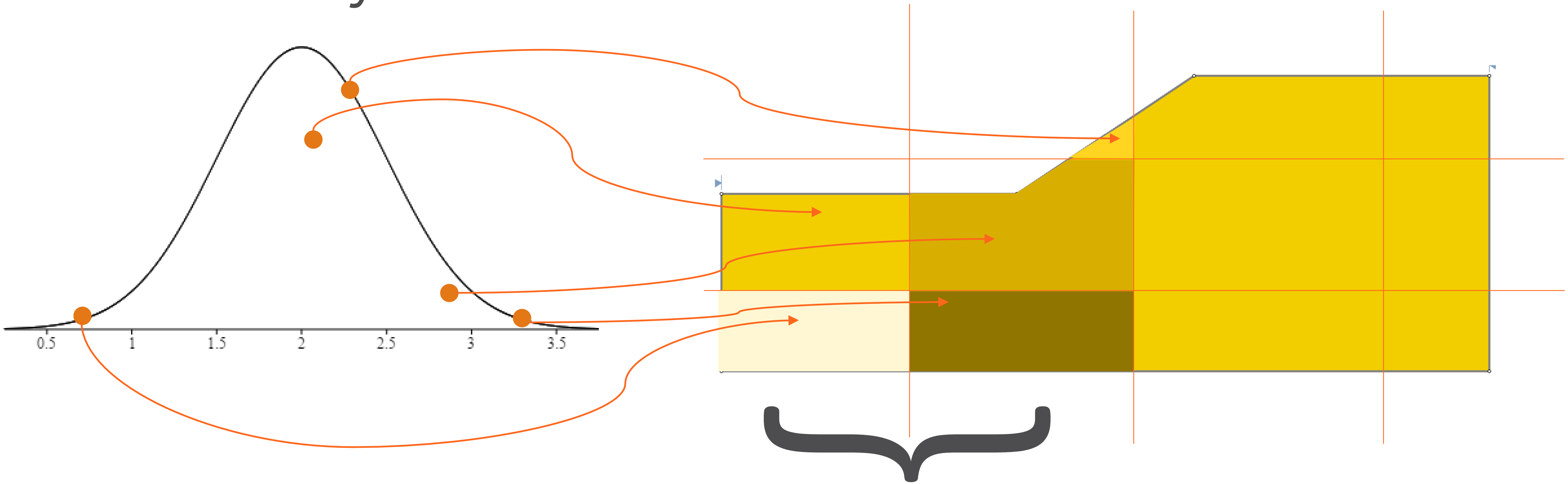
Simulation #1



Simulation #2



Variability *within* each simulation

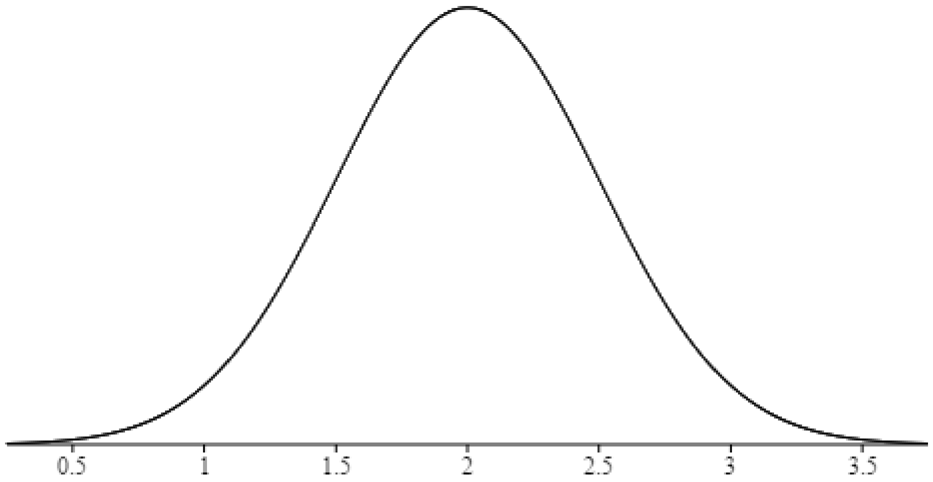


However, it is unlikely that a cohesion of 0.7 would change suddenly to 3.3 in nature.

Correlation Length

- **<u>Correlation Length, ϴ</u>** is the distance (in metres for example) over which the values of a random variable will be significantly correlated, or similar.

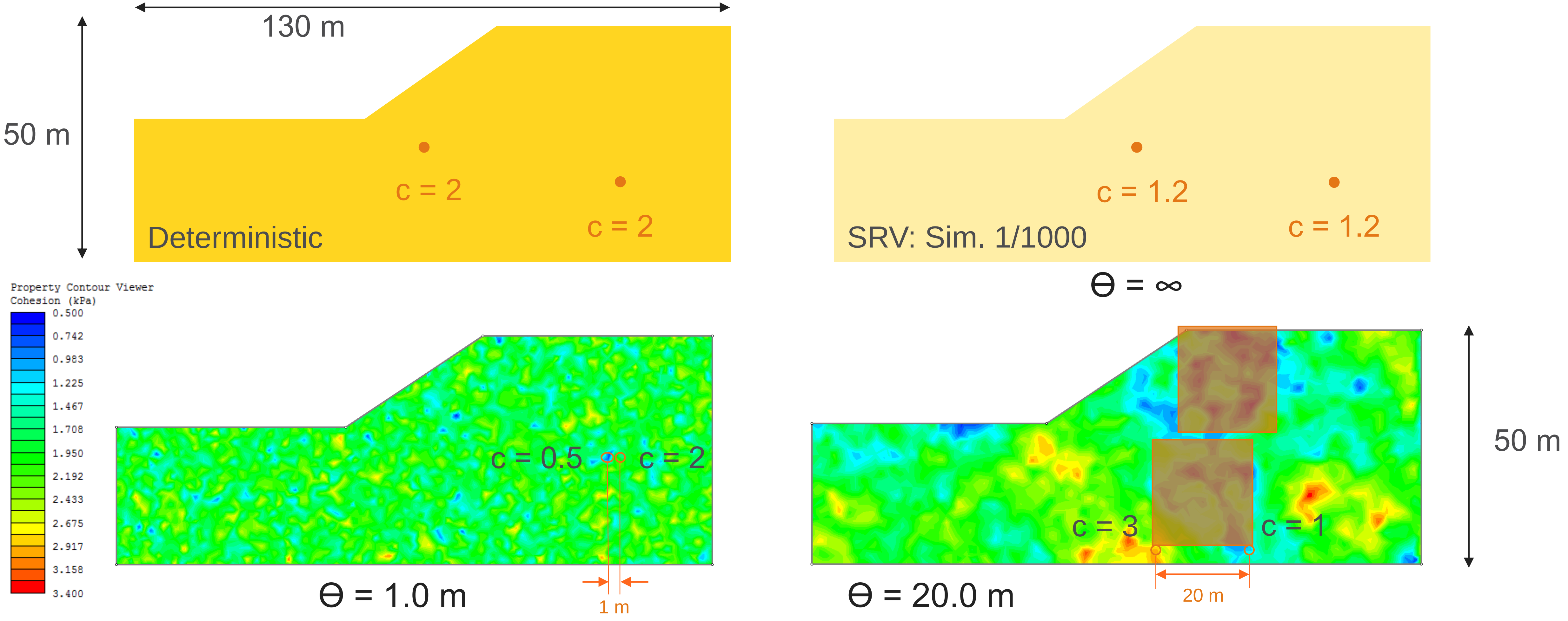
Correlation Length Visualization



Cohesion

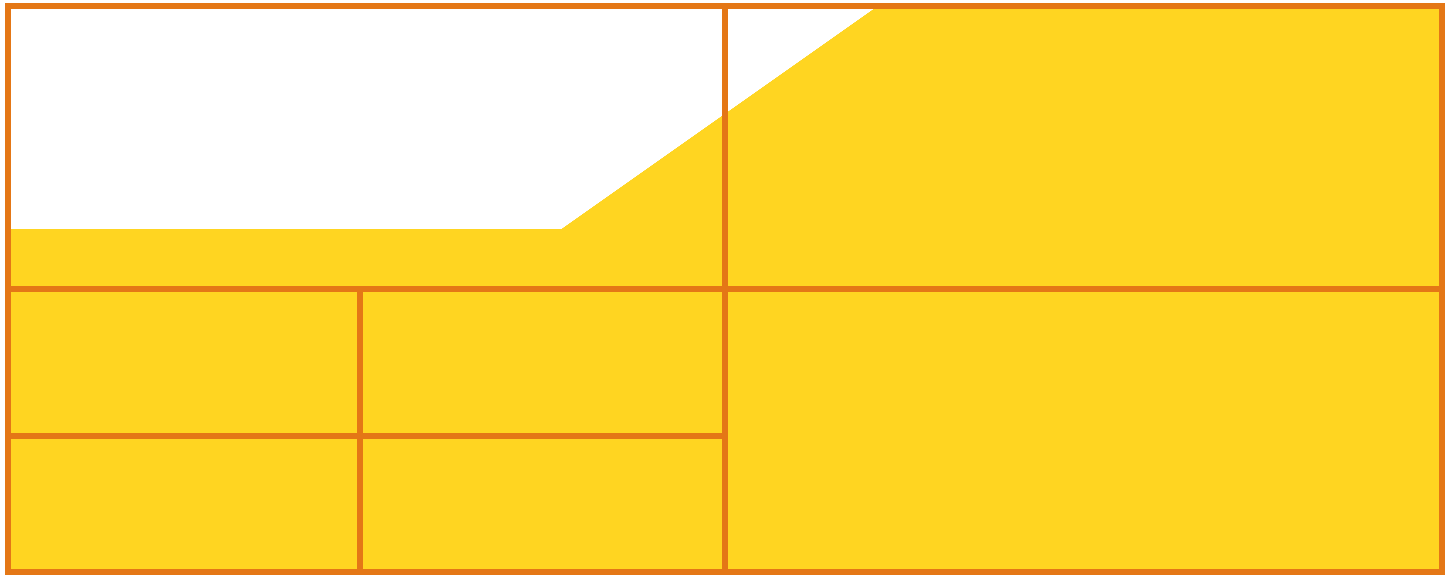
Correlation Length Visualization

Cohesion Parameters:
Mean: 2 kPa
Standard deviation: 0.5 kPa



Field Generation

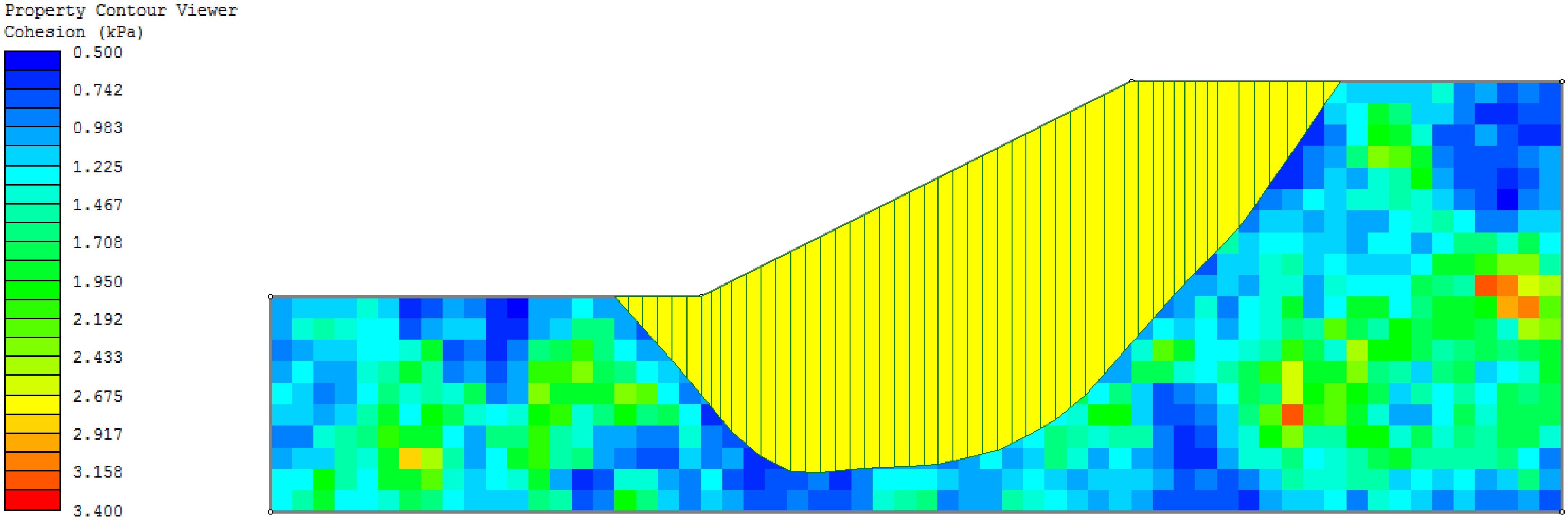
Random fields are generated using the Local Average Subdivision (LAS) method



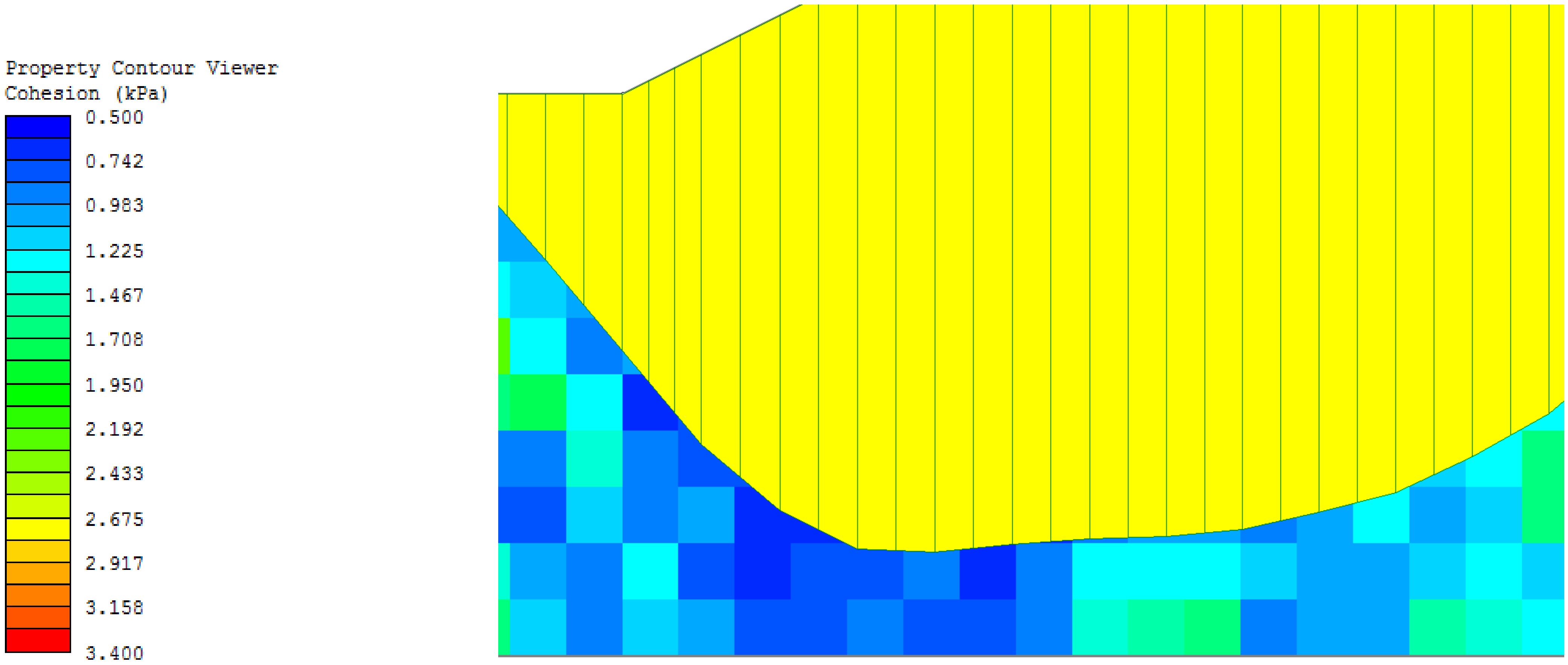
Field Generation

Random fields are generated using the Local Average Subdivision (LAS) method

Results: Simulation 1/1000

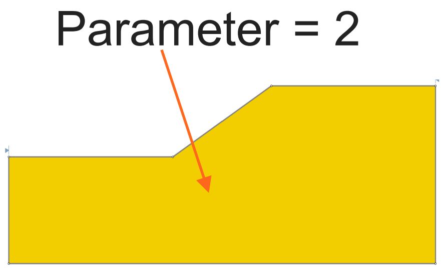


Results: Simulation 1/1000

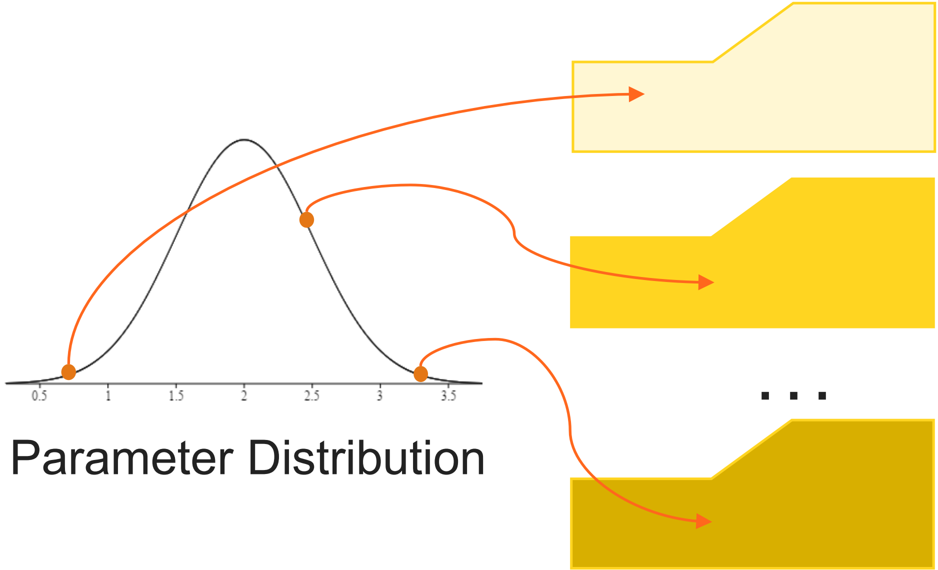


Deterministic to Probabilistic to Spatial

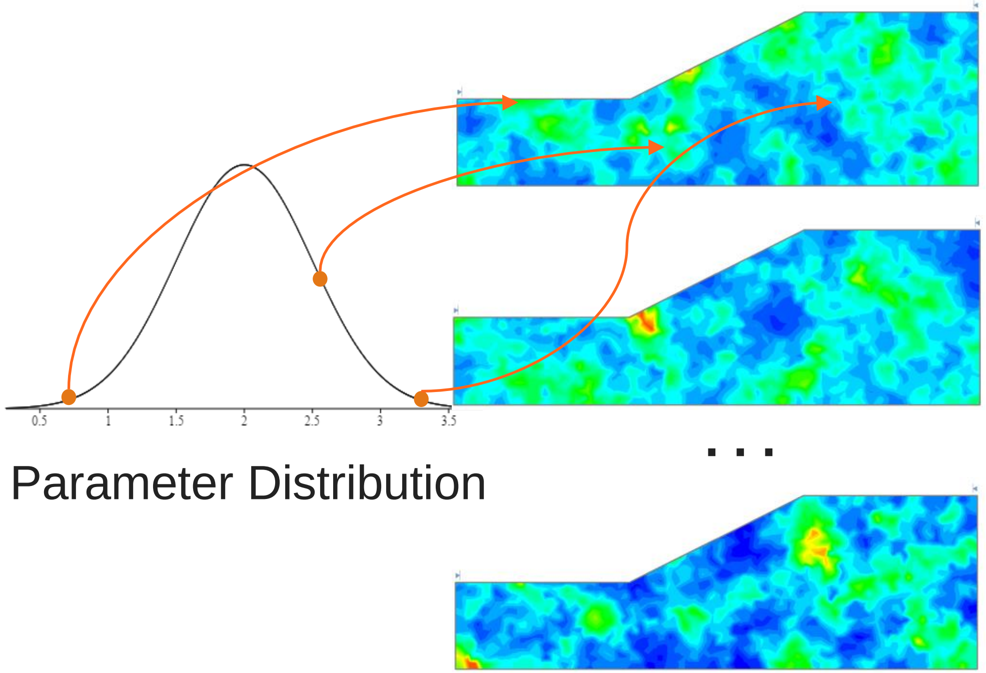
Deterministic analysis



Single Random Variable approach (SRV)



Random Limit Equilibrium Method (RLEM)



Spatial variability in Slide2 2018

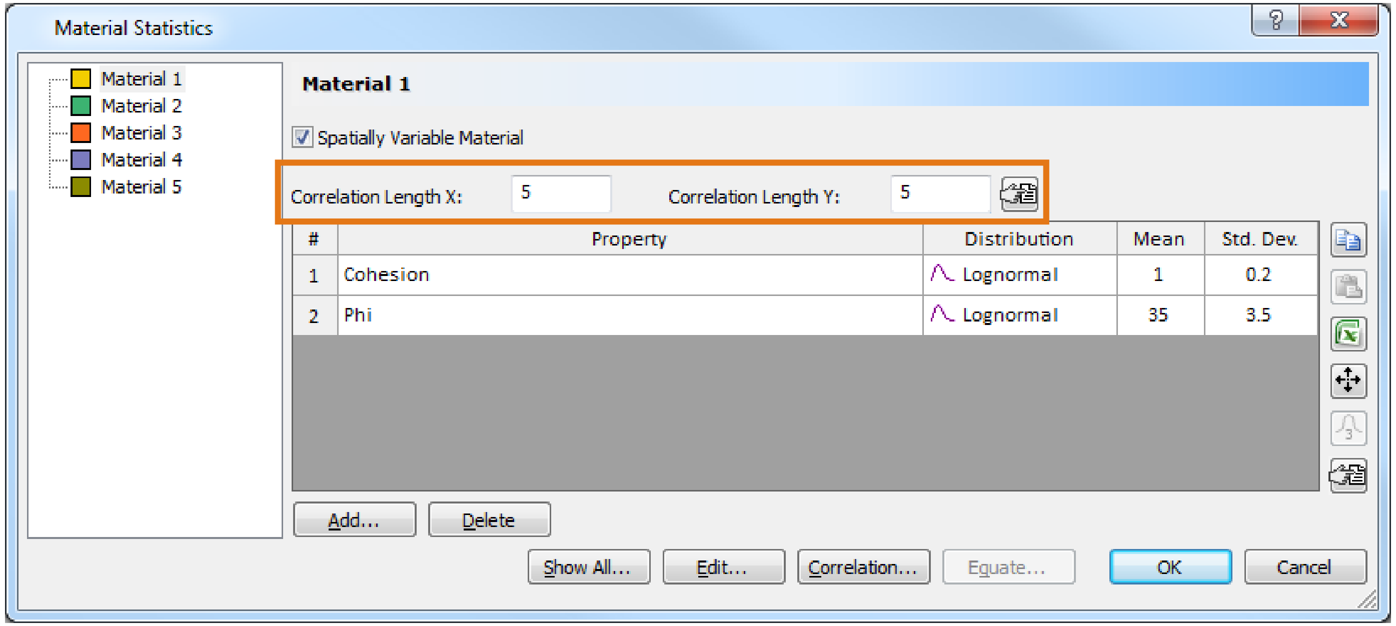
Section 5

Spatial Variability Analysis

Assign probability distributions to properties → Define correlation length per material → Generate n fields → Calculate FS for each simulation → Calculate PF

Spatial Variability Analysis

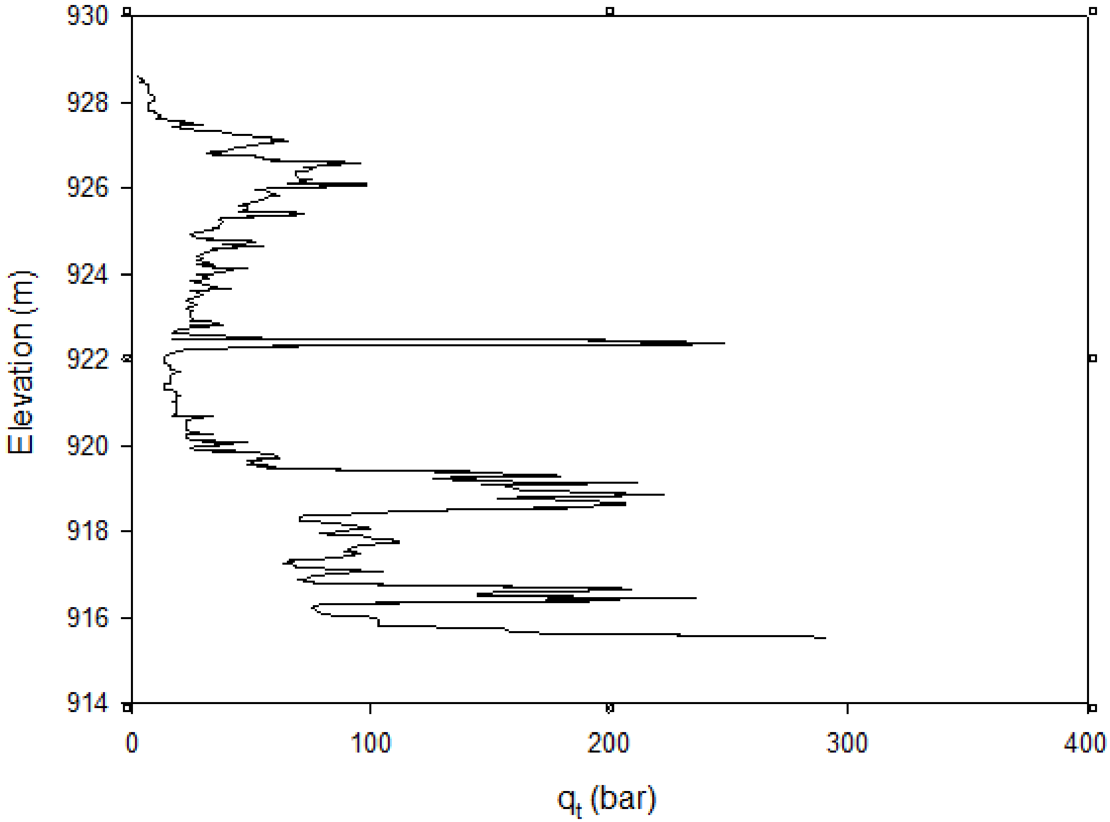
Assign probability distributions to properties → Define correlation length per material → Generate n fields → Calculate FS for each simulation → Calculate PF



Measuring Correlation Length

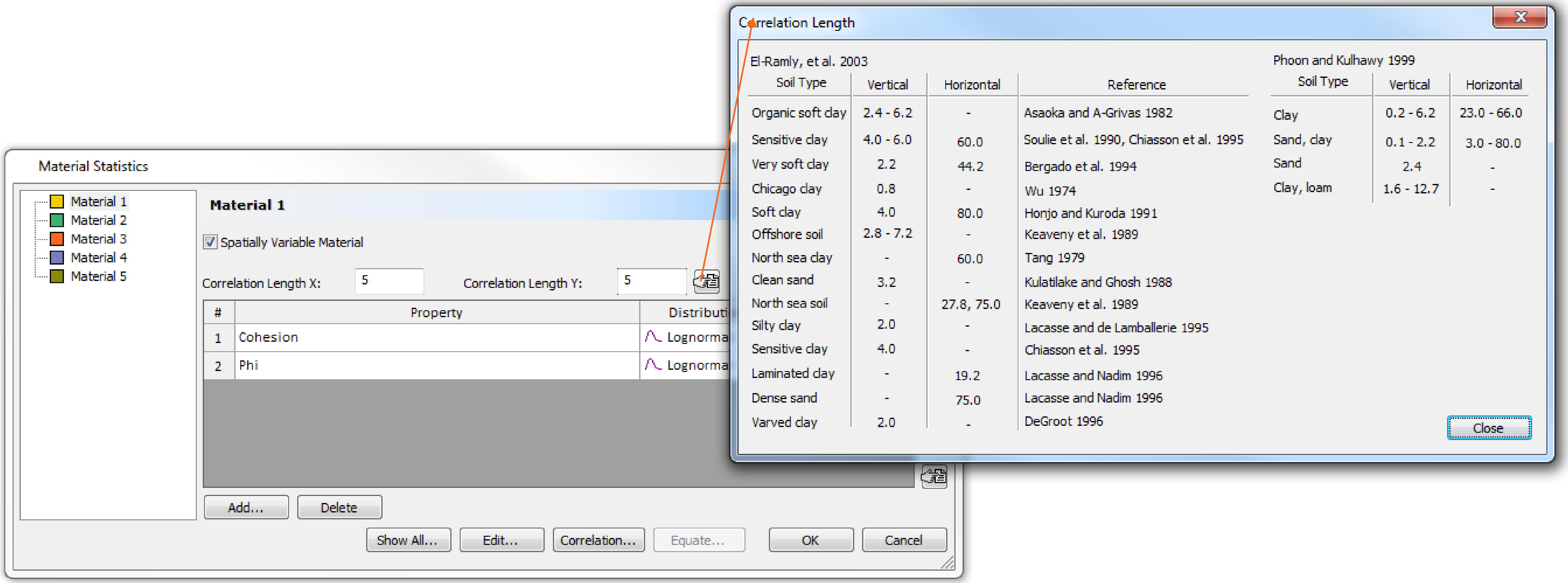
- CPT / SPT data
- Well-established method of measuring correlation length
- More advanced methods use Bayes to make up for insufficient data

$$\hat{\rho}(\tau_j) = \frac{1}{\sigma_R^2(k-j)} \sum_{i=1}^{k-j+1} (X_i - \mu_R)(X_{i+1} - \mu_R)$$



Tip resistance data from CPT. The vertical correlation length calculated for this case was 0.86m.

Correlation Length from Literature



El-Ramly, et al. 2003

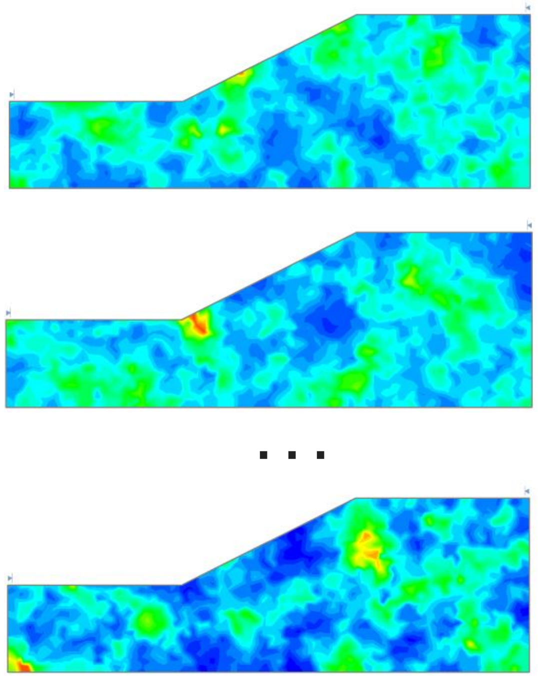
Soil Type	Vertical	Horizontal	Reference
Organic soft clay	2.4 - 6.2	-	Asaoka and A-Grivas 1982
Sensitive clay	4.0 - 6.0	60.0	Soulie et al. 1990, Chiasson et al. 1995
Very soft clay	2.2	44.2	Bergado et al. 1994
Chicago clay	0.8	-	Wu 1974
Soft clay	4.0	80.0	Honjo and Kuroda 1991
Offshore soil	2.8 - 7.2	-	Keaveny et al. 1989
North sea clay	-	60.0	Tang 1979
Clean sand	3.2	-	Kulatilake and Ghosh 1988
North sea soil	-	27.8, 75.0	Keaveny et al. 1989
Silty clay	2.0	-	Lacasse and de Lamballerie 1995
Sensitive clay	4.0	-	Chiasson et al. 1995
Laminated clay	-	19.2	Lacasse and Nadim 1996
Dense sand	-	75.0	Lacasse and Nadim 1996
Varved clay	2.0	-	DeGroot 1996

Phoon and Kulhawy 1999

Soil Type	Vertical	Horizontal
Clay	0.2 - 6.2	23.0 - 66.0
Sand, clay	0.1 - 2.2	3.0 - 80.0
Sand	2.4	-
Clay, loam	1.6 - 12.7	-

Spatial Variability Analysis

Assign probability distributions to properties → Define correlation length per material → Generate n fields → Calculate FS for each simulation → Calculate PF

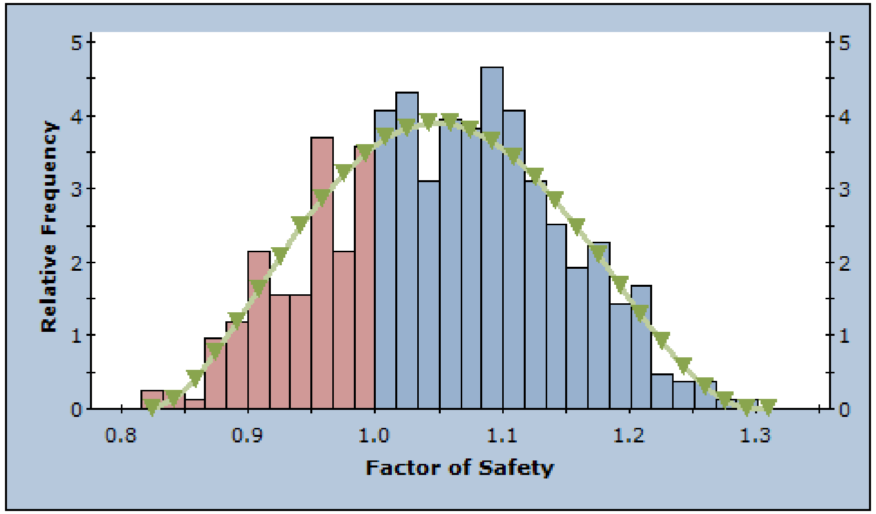


Spatial Variability Analysis

Assign probability distributions to properties → Define correlation length per material → Generate n fields → Calculate FS for each simulation → Calculate PF

$$PF = \frac{\#\ of\ simulations\ with\ FS < 1}{Total\ \#\ of\ simulations}$$

Random Limit Equilibrium Method (RLEM)



Probability of Failure: 0.292
BEST FIT: mean = 1.05226 fit = Beta s.d. = 0.0905499 min = 0.824479 max = 1.29539

Spatial variability in practice

Current spatial methods

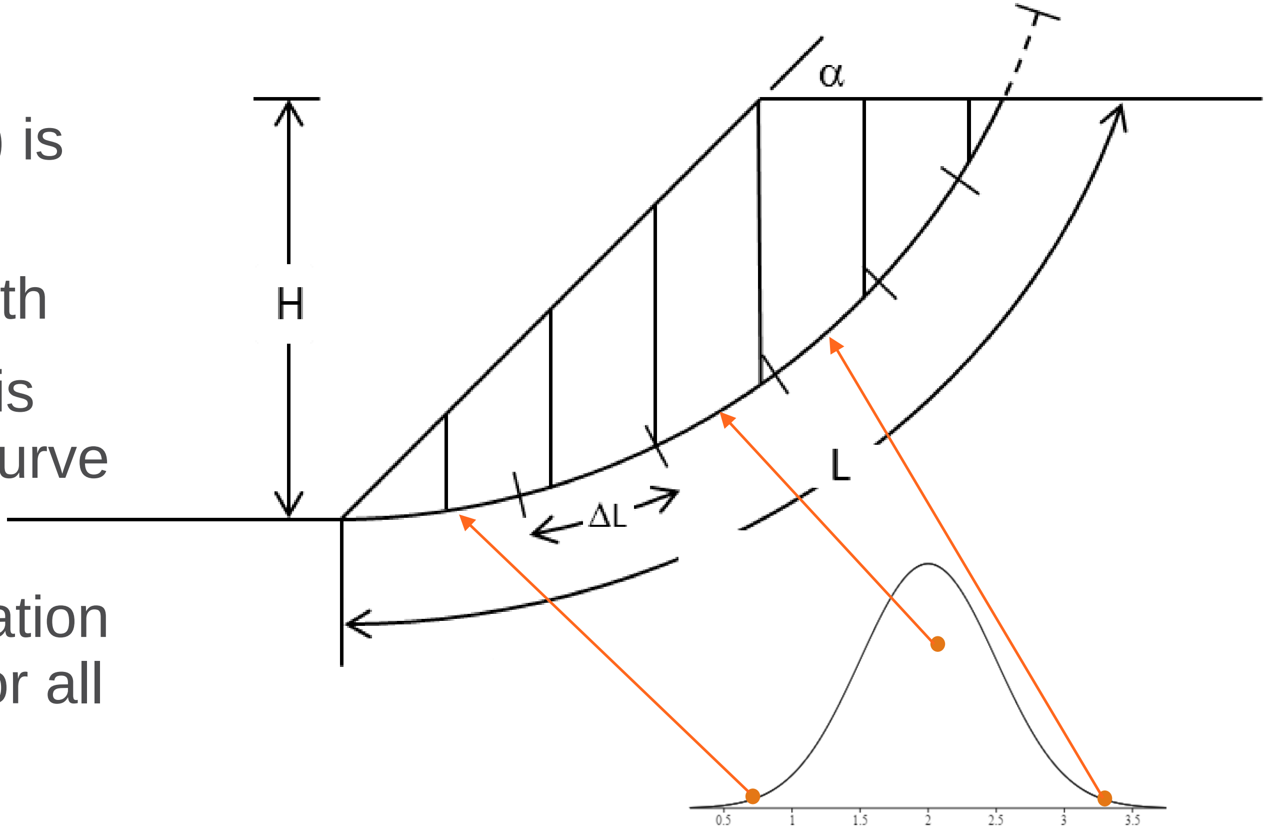
Method	Model	Publication	Year
1D spatial variability	James Bay dyke	H. El-Ramly, N.R. Morgenstern D.M. Cruden	2002
1D spatial variability	Sugar Creek embankment	S.E. Cho	2007
RFEM	Simple slopes	J. Huang D.V. Griffiths G.A. Fenton	2010
RLEM	Simple Slopes	Sina Javankhoshdel N. Luo R.J. Bathurst	2017
RLEM	Mount Polley dam	Brigid Cami Sina Javankhoshdel R.J. Bathurst	2017

1D Spatial variability

- Critical slip surface (length of L) is divided into equal segments

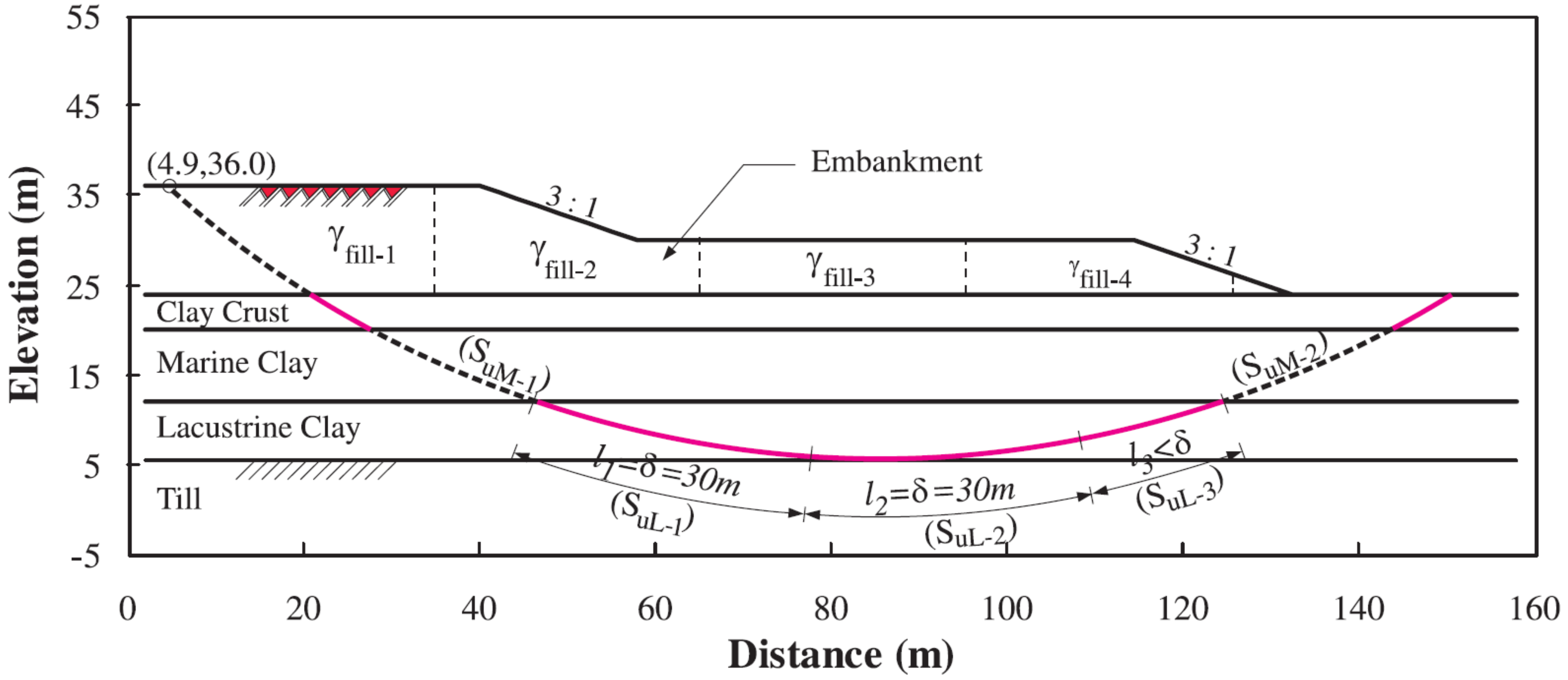
  ΔL = spatial correlation length

- For each segment one sample is assigned from the distribution curve of the parameter
- There is only one spatial correlation length (inaccurate), the same for all materials
- The fastest method, the least accuracy



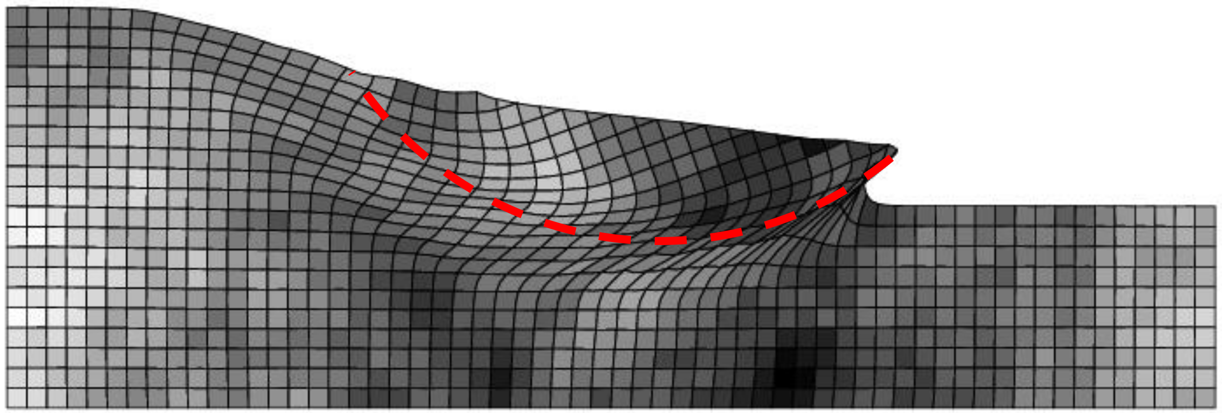
Parameter Distribution

1D Spatial variability: James Bay dyke



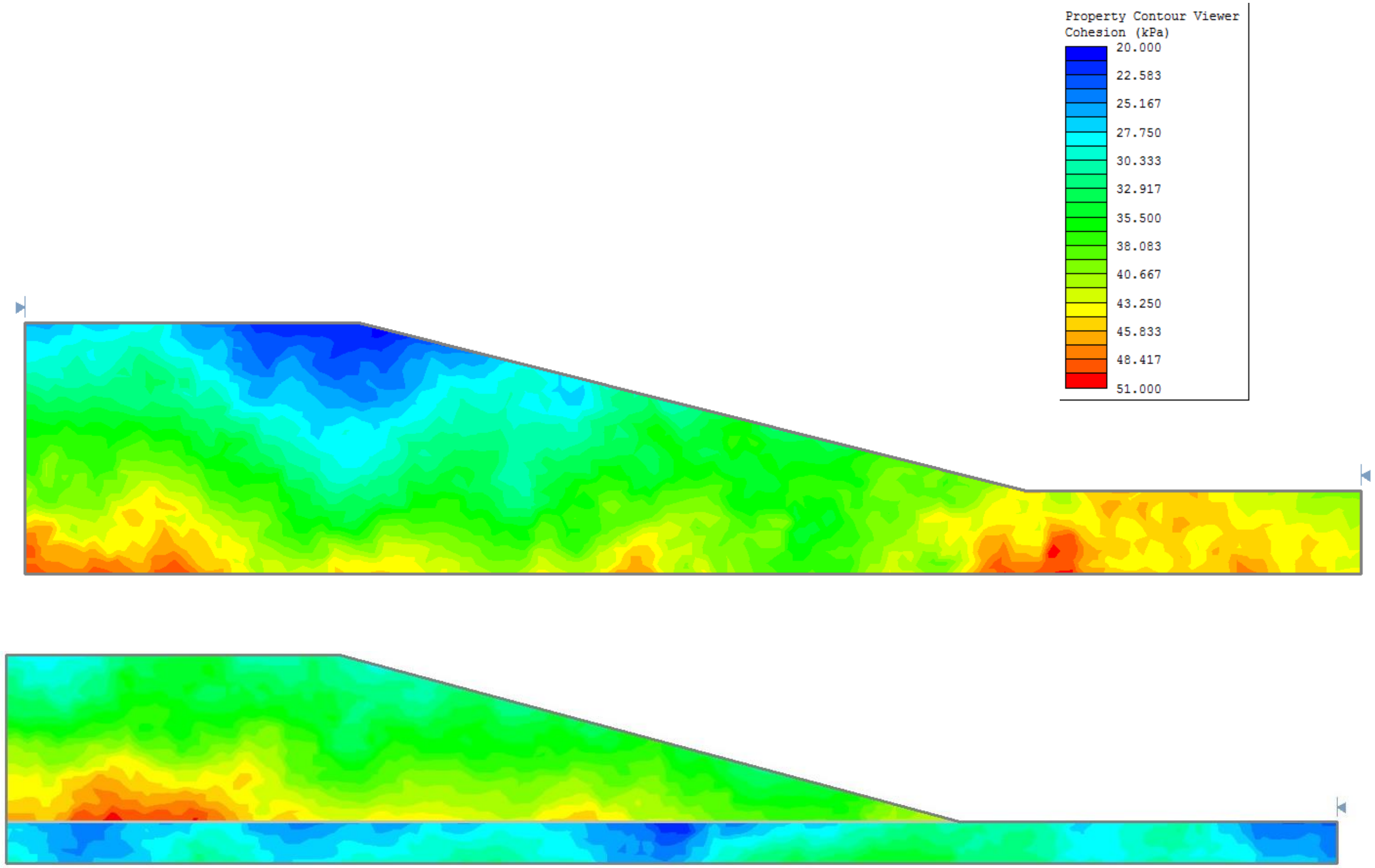
Random Finite Element Method (RFEM)

- Spatial field is created on a finite element mesh using LAS method
- Shear Strength Reduction (SSR) is used to calculate FS (deterministic part is independent from probabilistic part)
- Gravity turn-on method is used to calculate PF (fail and not fail with no Mean FS)
- The output will be Det. FS from SSR and PF from Gravity turn-on
- Speed problem, cannot handle complex geometries, cannot handle very small mesh sizes (convergence problem)



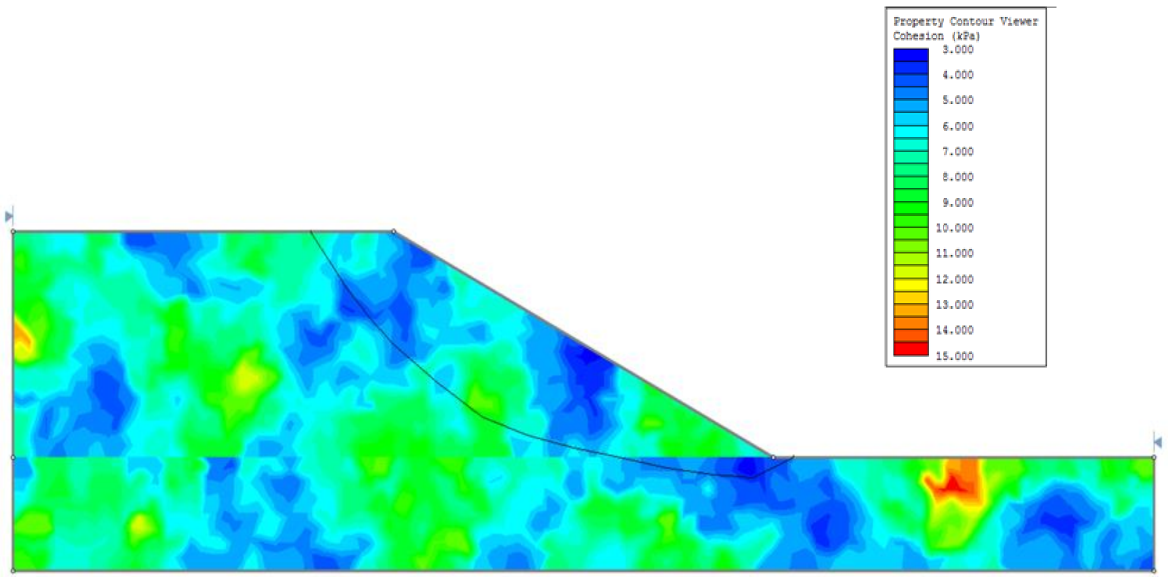
Random Limit Equilibrium Method (RLEM)

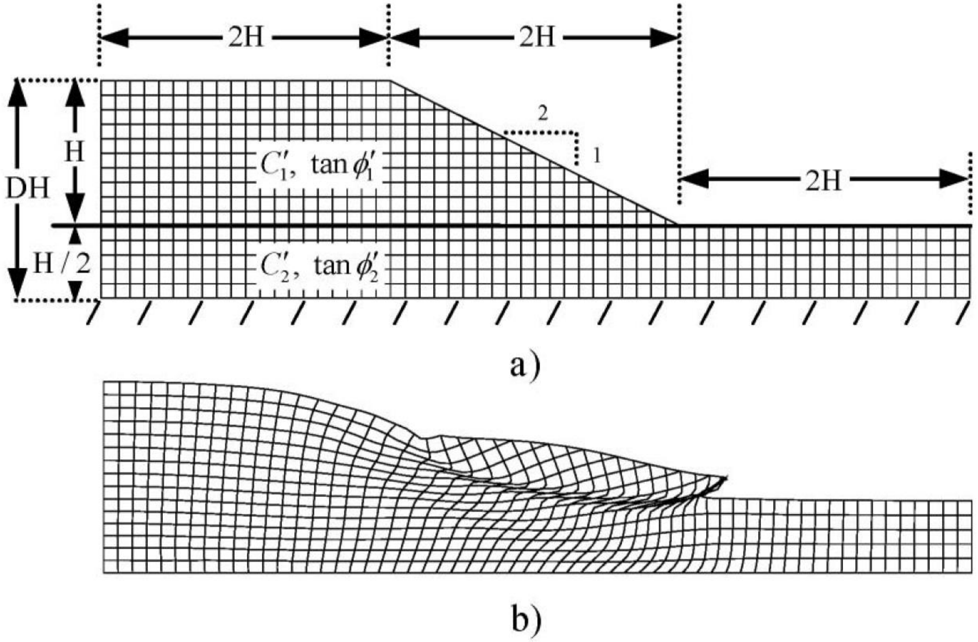
- Spatial field is created on a mesh using LAS method
- FS is calculated using non-circular LEM with optimization
- The output will be Det. FS, Mean FS and PF
- Any type of geometry can be considered
- Very fast, can detect the same failure surface as RFEM, any mesh size can be modeled (no convergence problem)



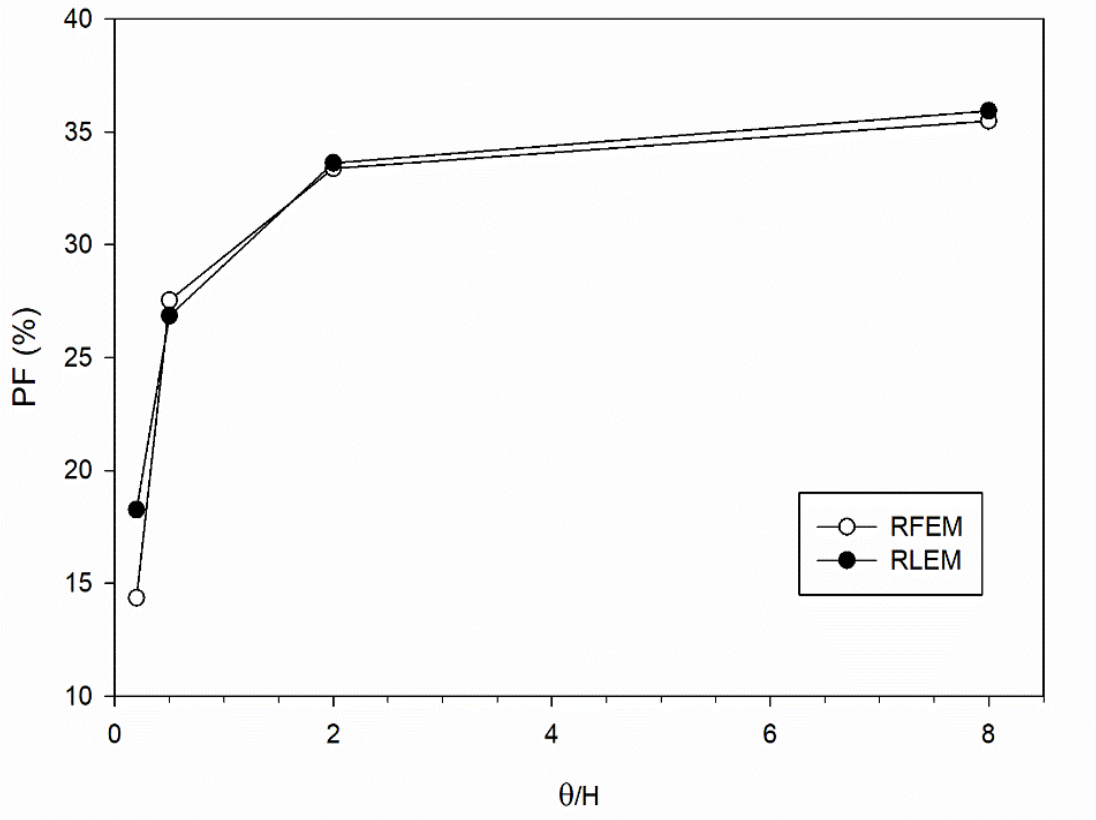
RLEM vs. RFEM

Parameter	Results
Factor of Safety	1.2
RFEM PF	10.5%
RLEM PF	14.5%

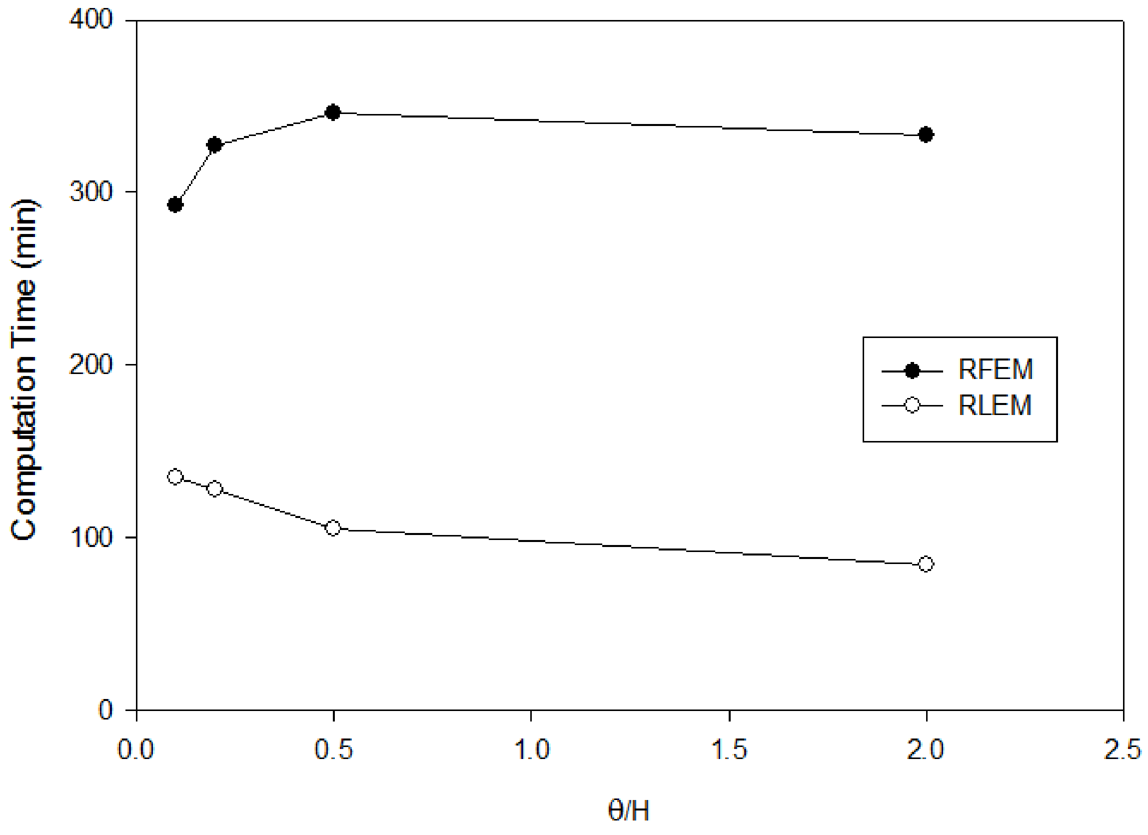




RLEM vs. RFEM: Computation time



5000 Monte Carlo simulations



Summary of current methods

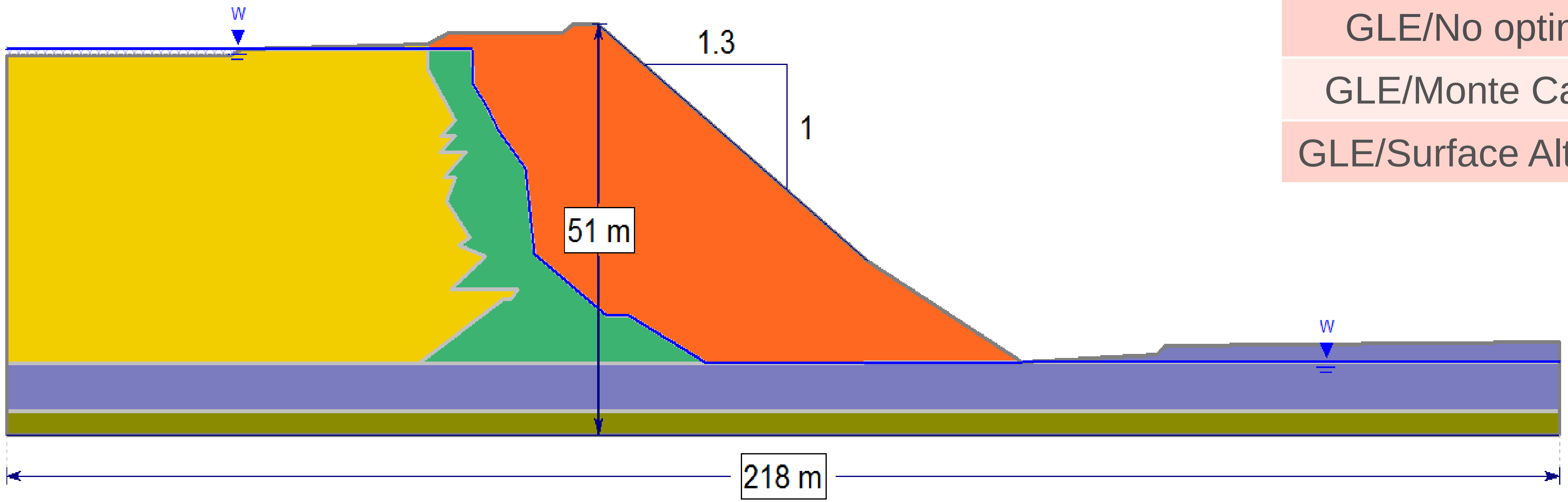
Method	Accuracy	Speed	Complex Geometries	Problem-Free Convergence (Mesh)
1D Spatial		✓	✓	N/A
RFEM	✓			
RLEM (*Slide2*)	✓	✓	✓	✓

List of publications

- Javankhoshdel, S., Luo, N. and Bathurst, R.J. 2017. Probabilistic analysis of simple slopes with cohesive soil strength using RLEM and RFEM. Georisk 11(3): 231-246.
- Cami, B., Javankhoshdel, S., Lam, J., Bathurst, R.J. and Yacoub, T. 2017. Probabilistic analysis of a tailings dam using 2D composite circular and non-circular deterministic analysis, SRV approach, and RLEM. 70th Canadian Geotechnical Conference, Ottawa, Ontario, 7 p.
- Javankhoshdel, S., Cami, B., Bathurst, R.J., Yacoub, T. and Corkum, B. 2017. Probabilistic analysis of cohesive-frictional slopes using the RLEM (circular and con-circular) and the RFEM. 70th Canadian Geotechnical Conference, Ottawa, Ontario, 8 p.
- Javankhoshdel, S., Cami, B., Bathurst, R.J., and Corkum, B. 2018. Probabilistic Analysis of Layered Slopes with Linearly Increasing Cohesive Strength and 2D Spatial Variability of Soil Strength Parameters Using Non-Circular RLEM Approach, IFCEE Conference, 133-142.
- Cami, B., Javankhoshdel, S., Lam, J., Bathurst, R.J. and Yacoub, T. 2018. Influence of Mesh Size, Number of Slices, and Number of Simulations in Probabilistic Analysis of Slopes Considering 2D Spatial Variability of Soil Properties, IFCEE Conference, 186-196.

Spatial Examples with *Slide2*

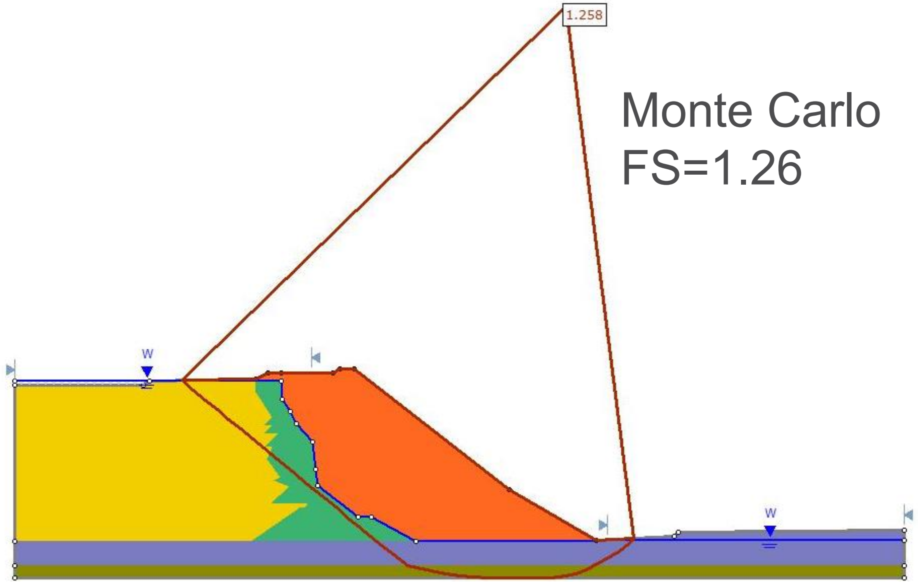
Example 1: Mount Polley tailings dam



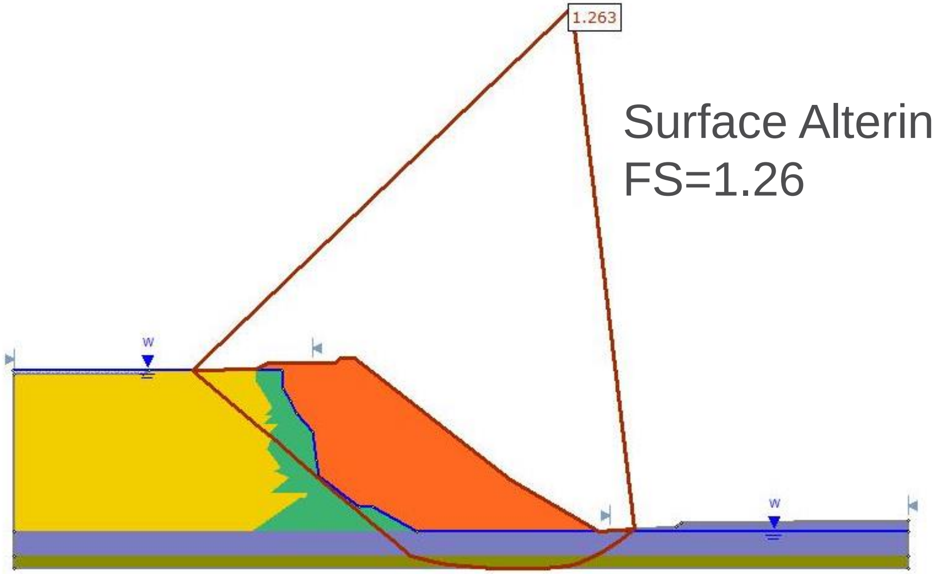
Method
GLE/No optimization
GLE/Monte Carlo (MC)
GLE/Surface Altering (SA)

Deterministic analysis

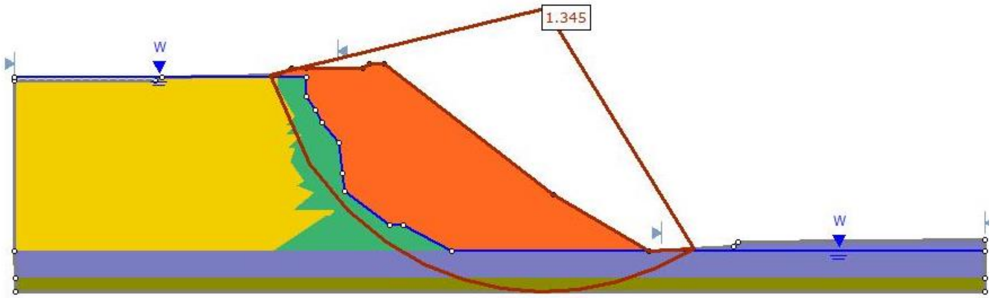
Optimization	Factor of Safety
No optimization	1.35
Monte Carlo (MC)	1.26
Surface Altering (SA)	1.26



Monte Carlo
FS=1.26

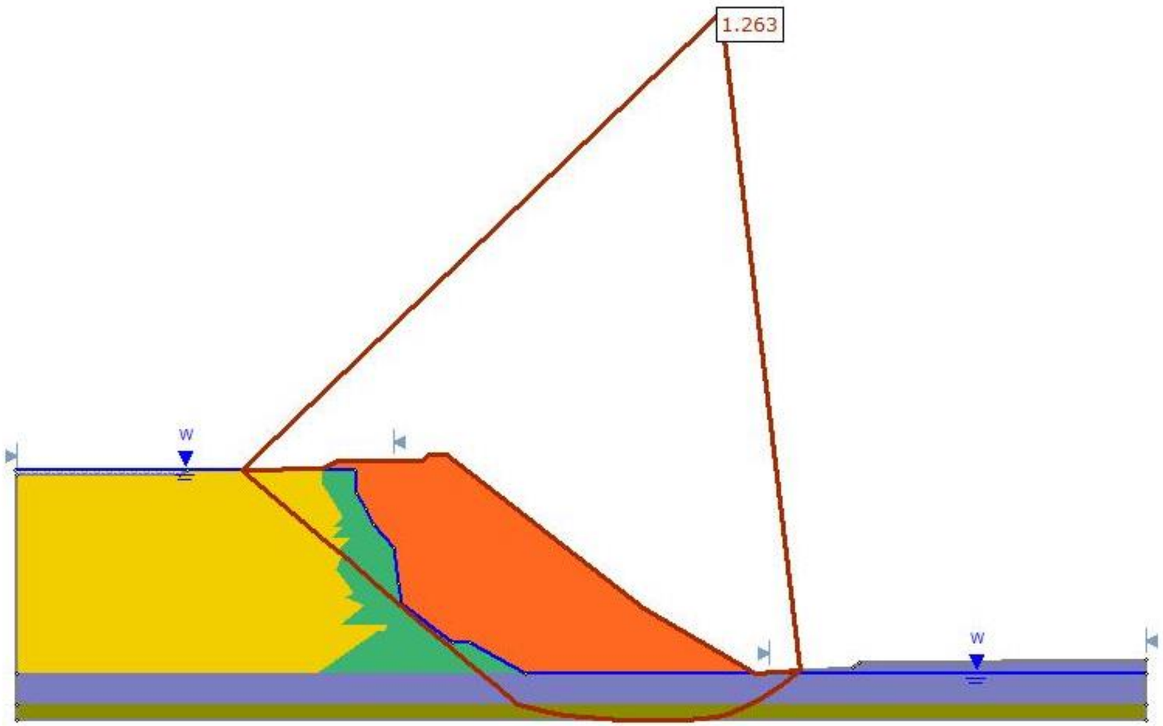


Surface Altering
FS=1.26

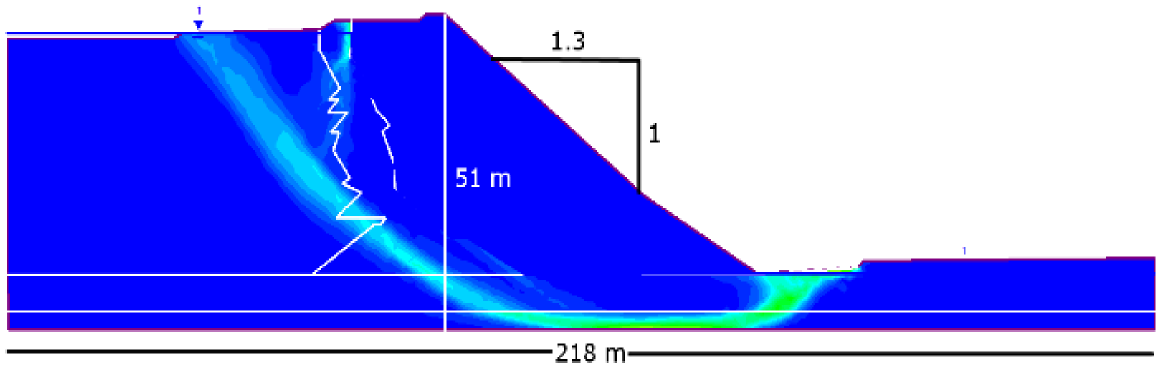


No optimization FS=1.35

Verify with SSR (*RS2*)



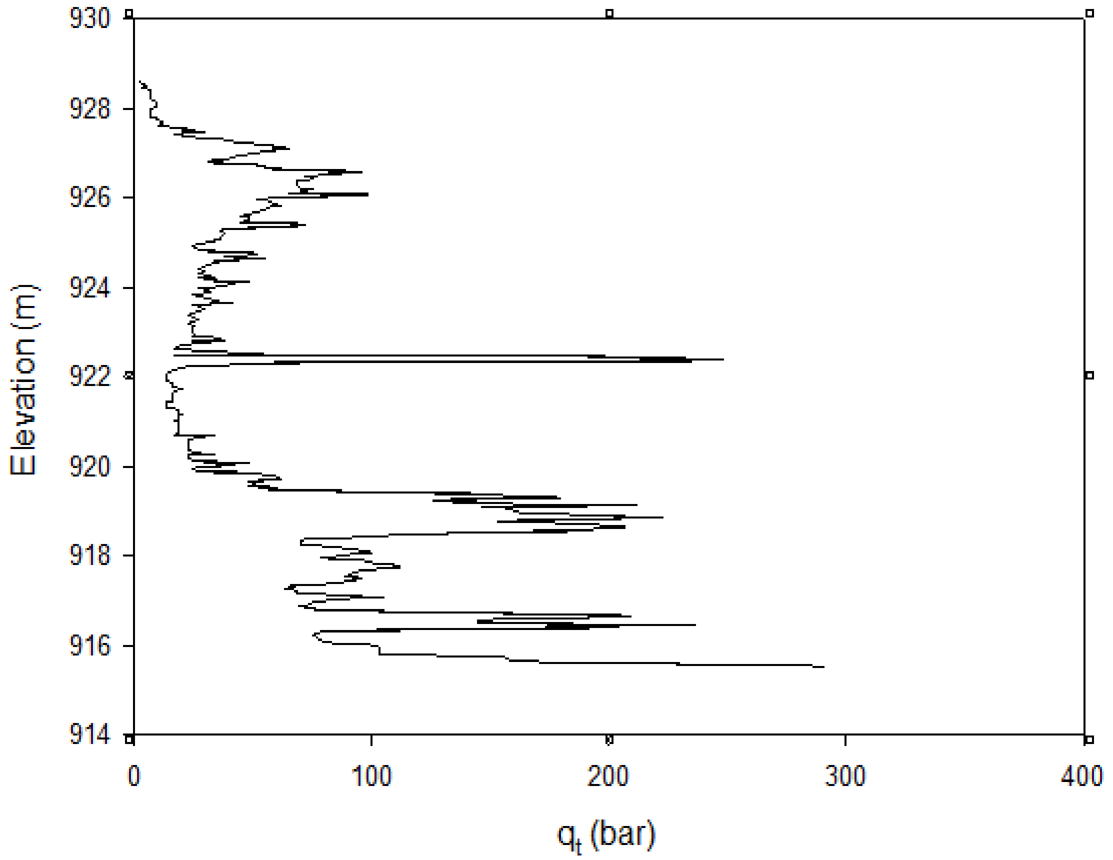
FS = 1.26



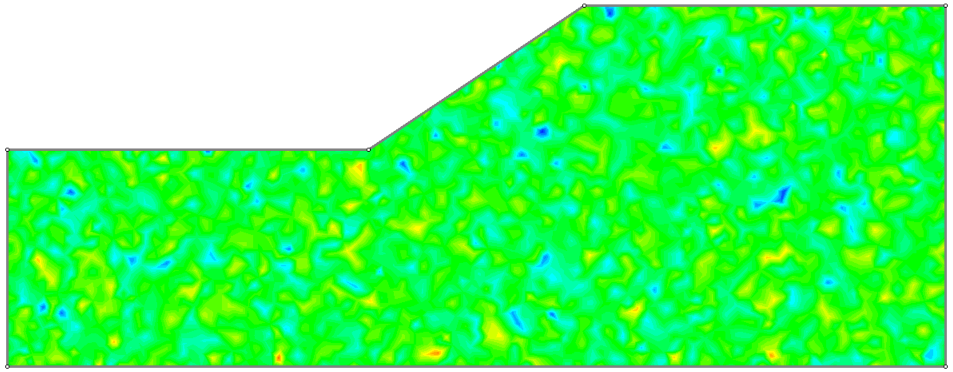
SRF = 1.26

Measuring spatial correlation length

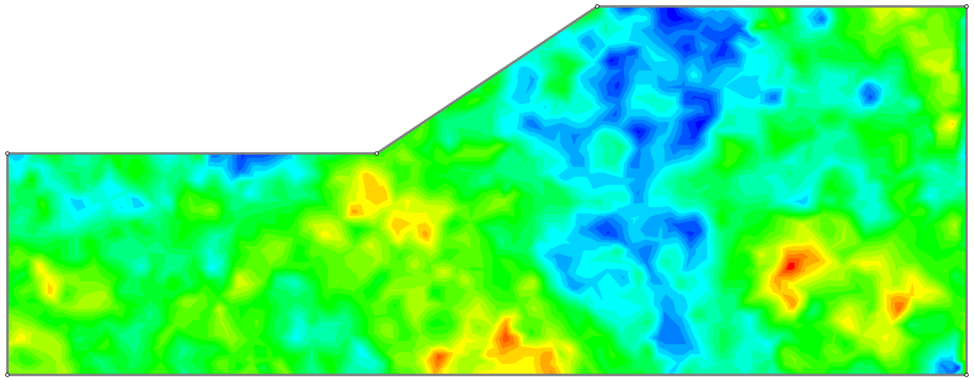
- Correlation length was measured at nine different CPT locations.
- Usually there is not enough CPT data to measure the horizontal spatial correlation length.
- A vertical correlation length of 1m and a horizontal spatial correlation length of 400m was used.



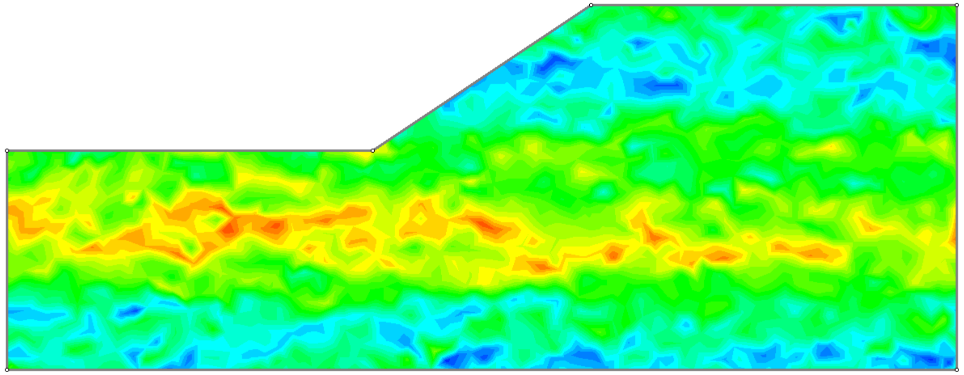
Correlation length visualization



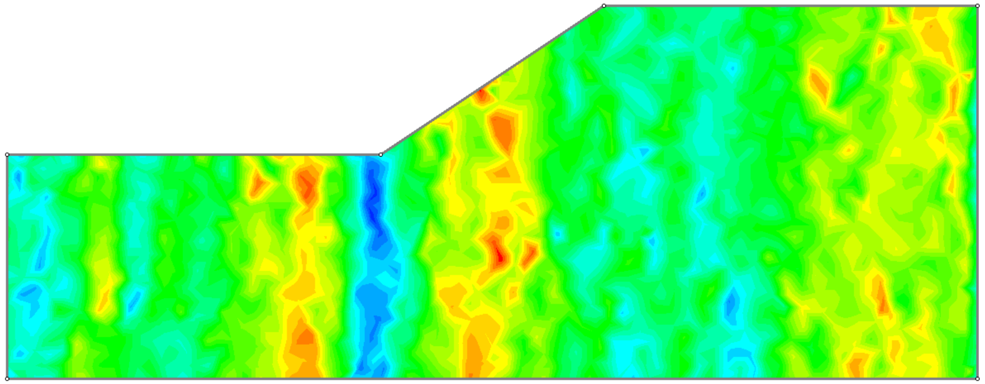
$\theta = 1$ m



$\theta = 20$ m



$\theta_H = 1000$ m, $\theta_V = 1$ m



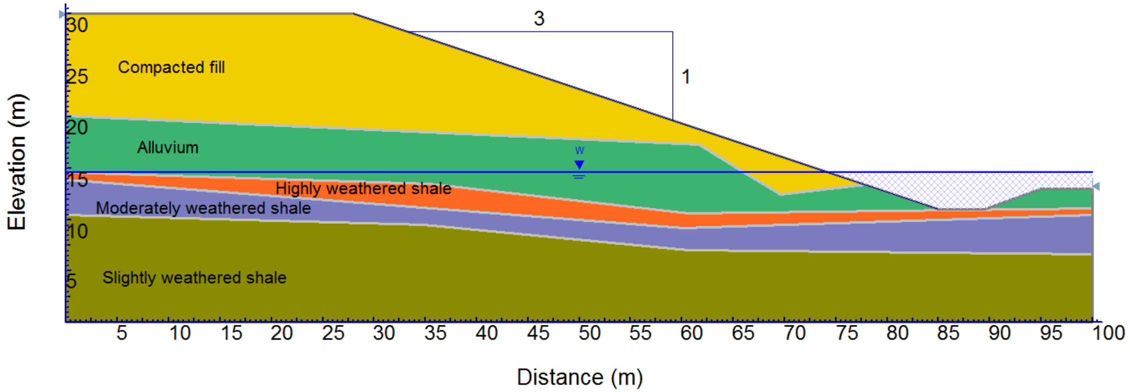
$\theta_H = 1$ m, $\theta_V = 1000$ m

Results

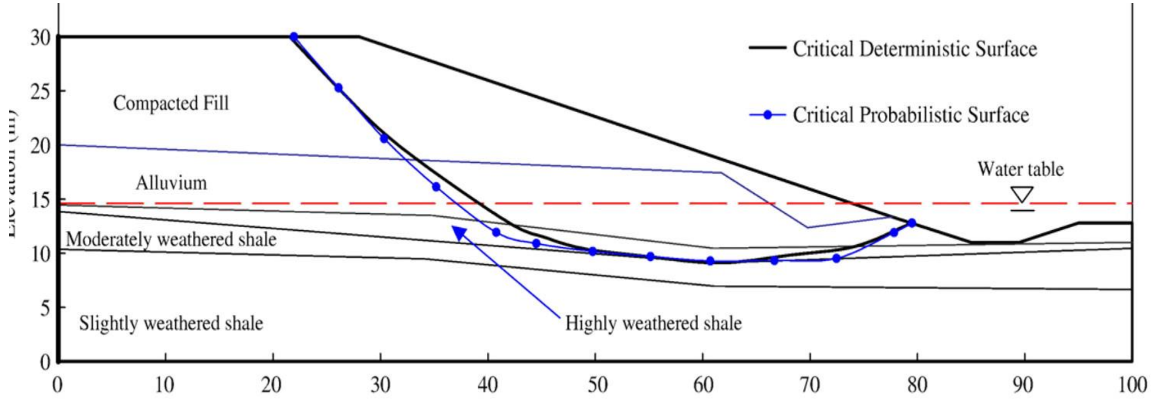
Parameter/Opt.	No Opt.	MCO	SAO
Det. FS	1.35	1.26	1.26
Mean FS	1.31	1.2	1.19
PF (%)	0	0.09	0.18
Simulation time (hours)*	1.5	22	4.5

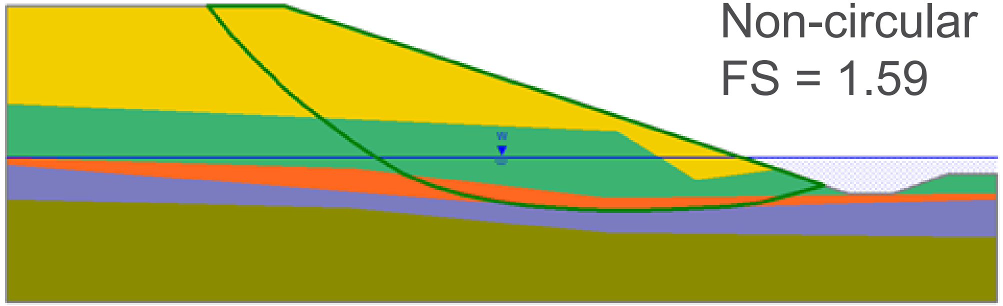
*for 10,000 simulations

Example 3: Sugar Creek

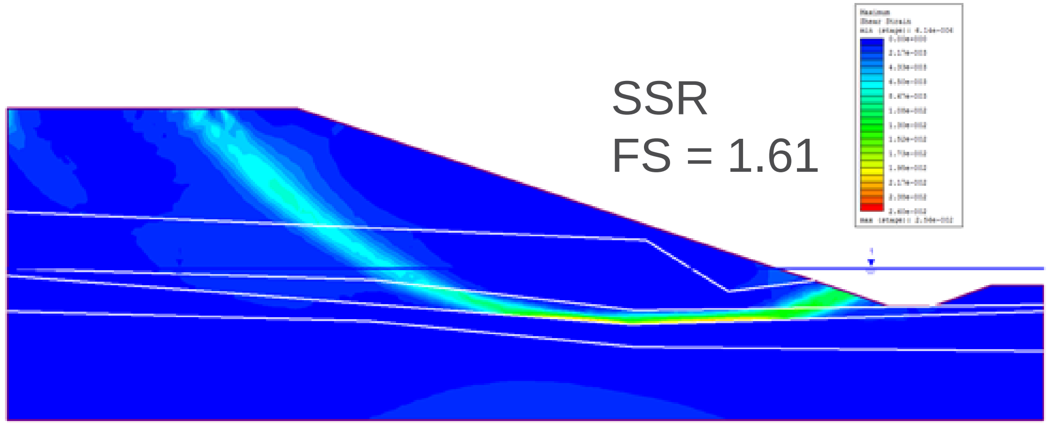


Non-circular
FS = 1.59





Non-circular
FS = 1.59

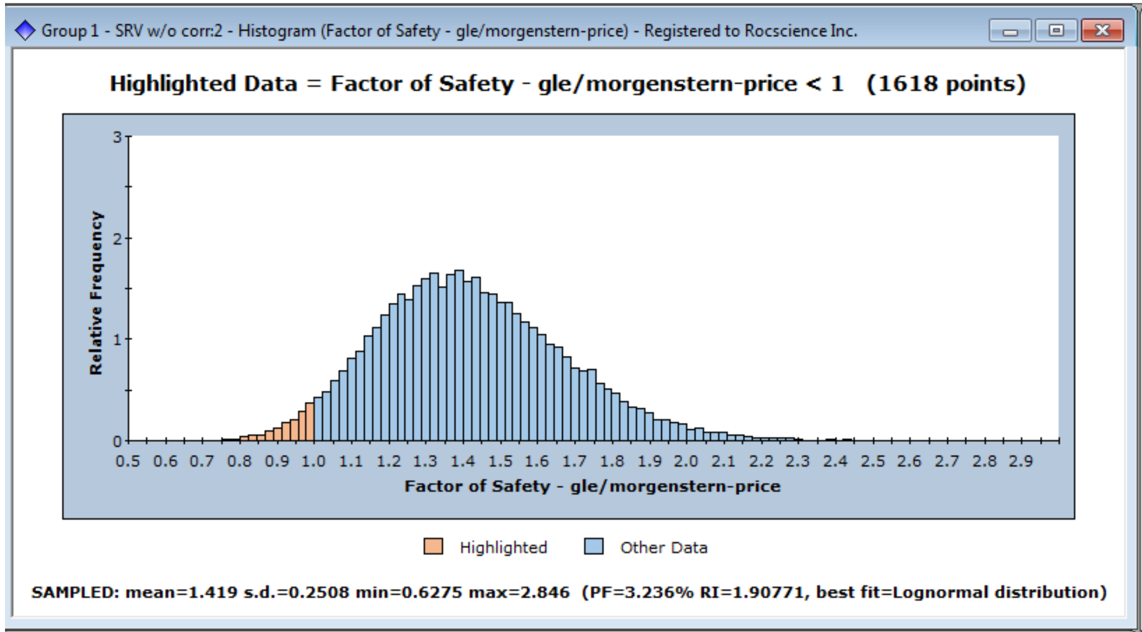


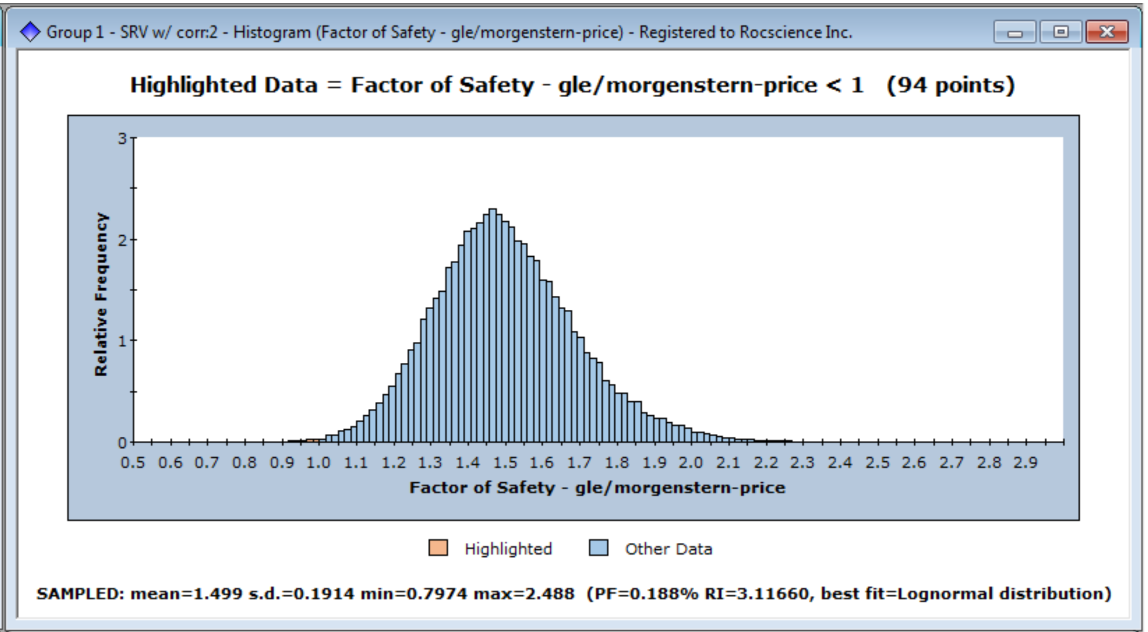
SSR
FS = 1.61

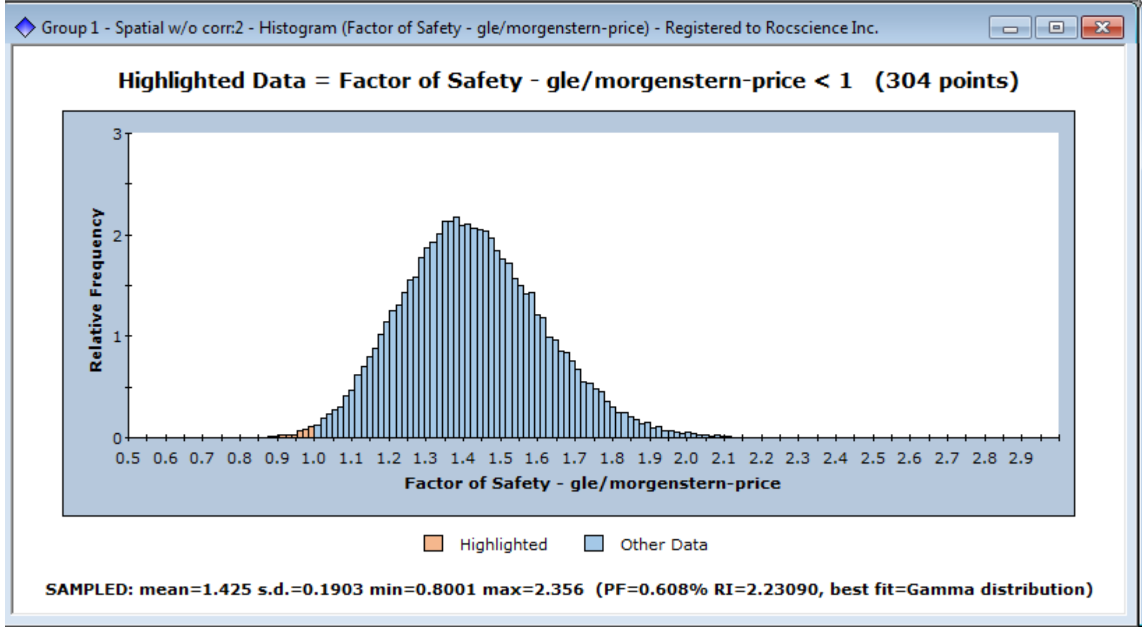
Sugar Creek results

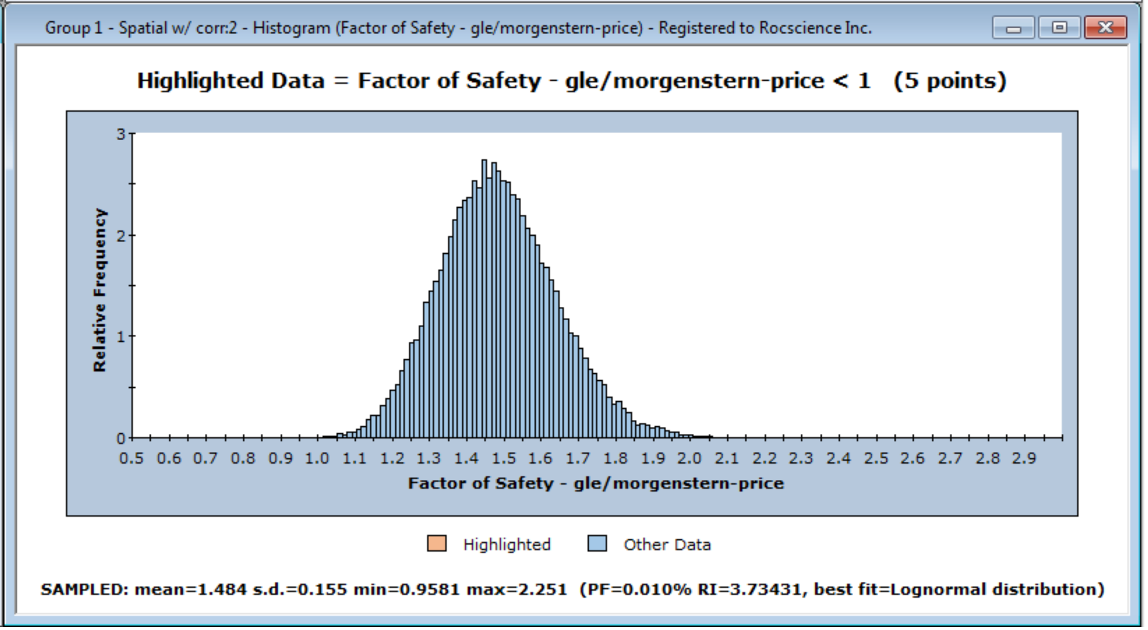
Method	PF without Cross-Correlation	PF with Cross-Correlation
SRV	3.24%	0.19%
RLEM	0.61%	0.01%

Deterministic FS: 1.59

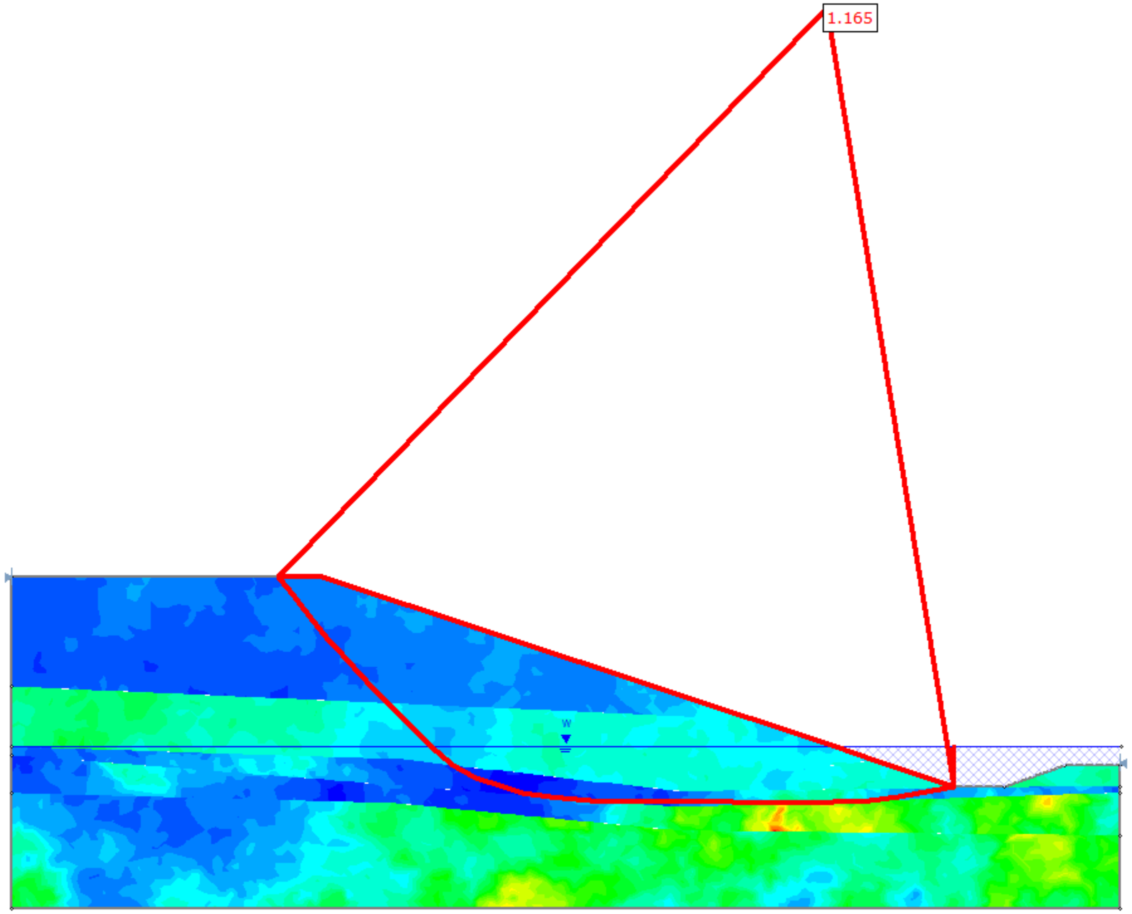

Group 1 - SRV w/o corr:2 - Histogram (Factor of Safety - gle/morgenstern-price) - Registered to Rocscience Inc.
Highlighted Data = Factor of Safety - gle/morgenstern-price < 1 (1618 points)
Relative Frequency
Factor of Safety - gle/morgenstern-price
Highlighted
Other Data
SAMPLED: mean=1.419 s.d.=0.2508 min=0.6275 max=2.846 (PF=3.236% RI=1.90771, best fit=Lognormal distribution)

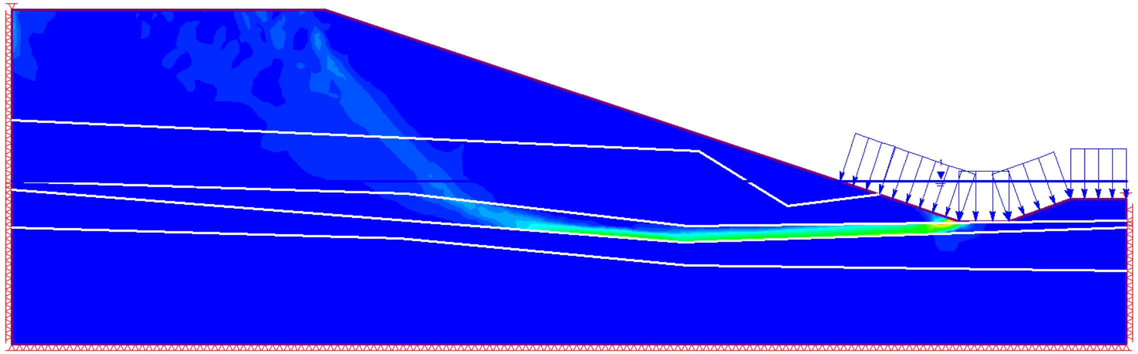

Group 1 - SRV w/ corr:2 - Histogram (Factor of Safety - gle/morgenstern-price) - Registered to Rocscience Inc.
Highlighted Data = Factor of Safety - gle/morgenstern-price < 1 (94 points)
Relative Frequency
Factor of Safety - gle/morgenstern-price
Highlighted
Other Data
SAMPLED: mean=1.499 s.d.=0.1914 min=0.7974 max=2.488 (PF=0.188% RI=3.11660, best fit=Lognormal distribution)


Group 1 - Spatial w/o corr:2 - Histogram (Factor of Safety - gle/morgenstern-price) - Registered to Rocscience Inc.
Highlighted Data = Factor of Safety - gle/morgenstern-price < 1 (304 points)
Relative Frequency
Factor of Safety - gle/morgenstern-price
Highlighted
Other Data
SAMPLED: mean=1.425 s.d.=0.1903 min=0.8001 max=2.356 (PF=0.608% RI=2.23090, best fit=Gamma distribution)

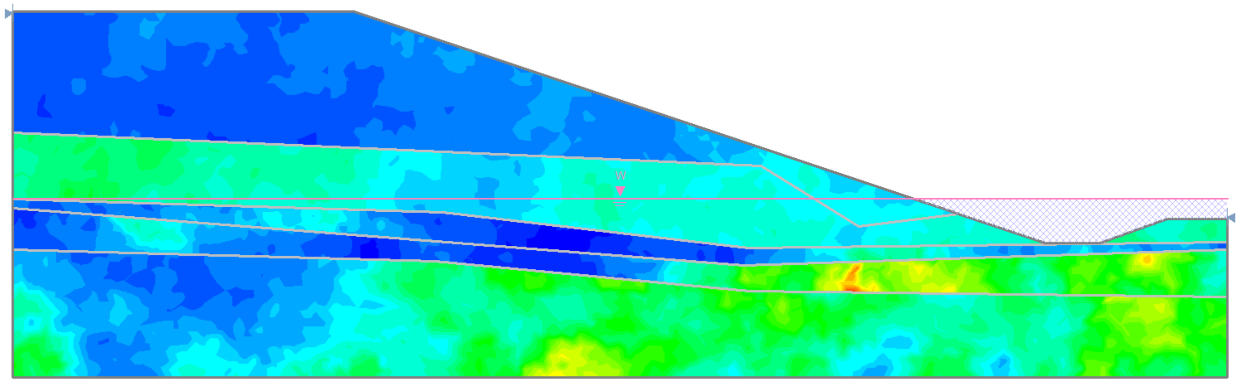

Group 1 - Spatial w/ corr:2 - Histogram (Factor of Safety - gle/morgenstern-price) - Registered to Rocscience Inc.
Highlighted Data = Factor of Safety - gle/morgenstern-price < 1 (5 points)
Relative Frequency
Factor of Safety - gle/morgenstern-price
Highlighted
Other Data
SAMPLED: mean=1.484 s.d.=0.155 min=0.9581 max=2.251 (PF=0.010% RI=3.73431, best fit=Lognormal distribution)

Slide2 and *RS2*

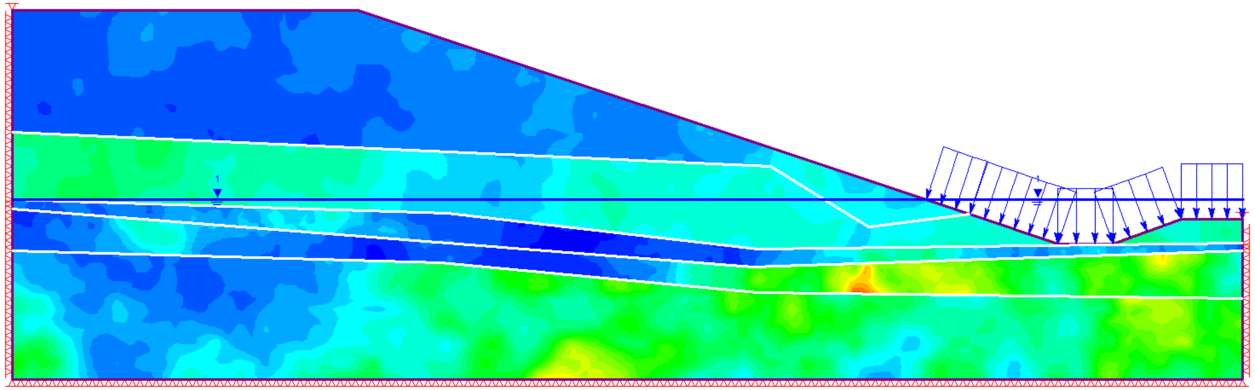




Slide2 and *RS2*



Slide2



RS2

Spatial variability in Tunneling

Section 6

Spatial variability in Tunneling

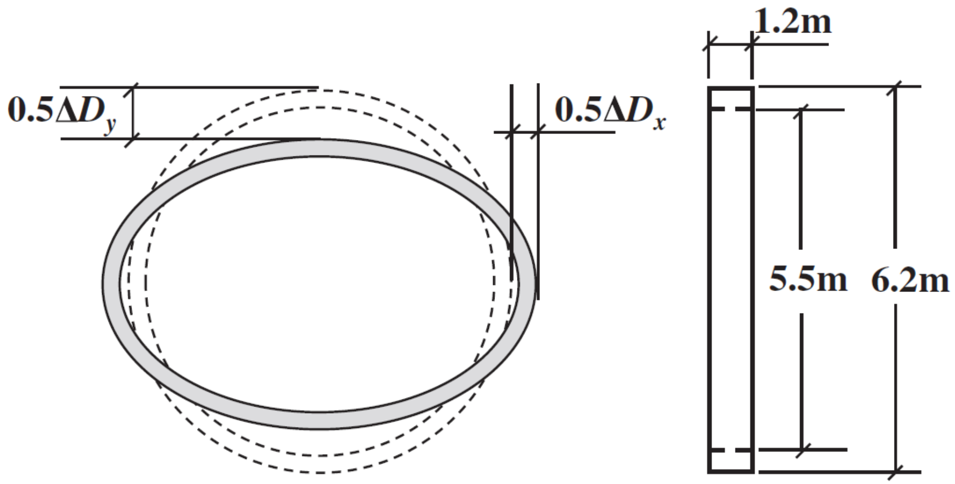
- **Huang et al (2017):** Influence of spatial variability of soil Young's modulus on tunnel convergence in soft soils.
- **Huang et al. (2015)**: Presenting that the longitudinal performance of shield tunnels in terms of differential settlement is significantly affected by the spatial variation of subgrade reaction coefficient in longitudinal direction.
- **Mollon et al. (2011)**: Analyzing the face stability of a tunnel driven in anisotropic and nonhomogeneous soils by considering the spatial variability of soil shear strength.

Example 1: Case study: Huang et al. (2017)

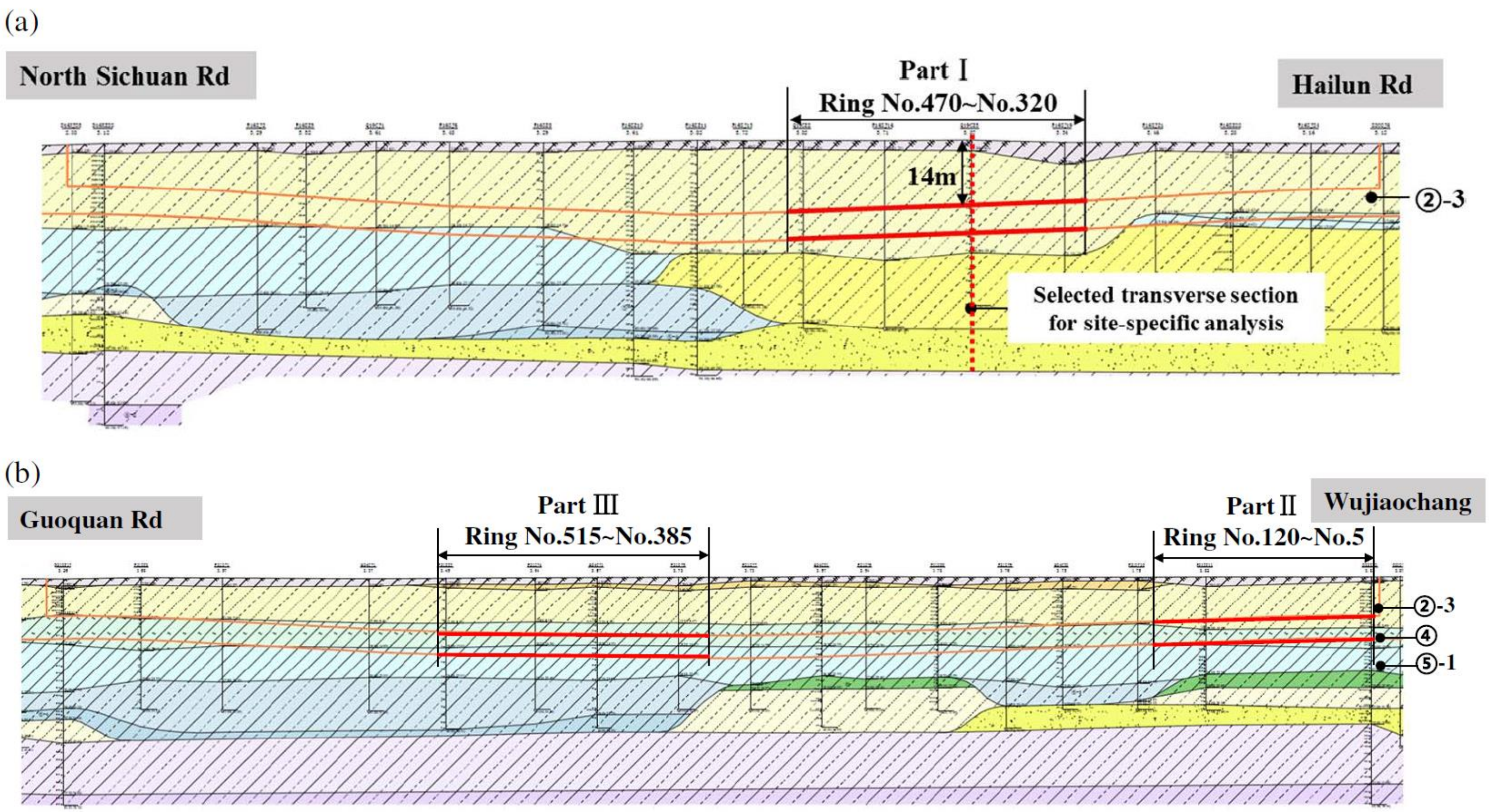
- Influence of spatial variability of soil Young's modulus on tunnel convergence in soft soils.
- Field data have shown the fact that soil spatial variability could aggravate the uncertainty of tunnel convergence ΔD (a key indicator for serviceability and safety of tunnels).
- Among all soil properties, it is widely accepted that the Young’s modulus E_s and Poisson's ratio ν_s are the dominant parameters which greatly affect deformations of soils and embedded geo-structures, e.g. tunnel lining convergence

Problem definition

- Since the convergence of tunnel lining is regarded to be a significant indicator of tunnel deformational performance, we investigate the influence of scale of fluctuation (SOF) of the soil Young's modulus on the convergence of tunnel lining in layered anisotropic soft ground.
- Both the isotropic and layered anisotropic random field for soil Young's modulus E_s is simulated.



Longitudinal geological profile of Shanghai metro line



Tip resistance from representative CPT tests obtained at Shanghai

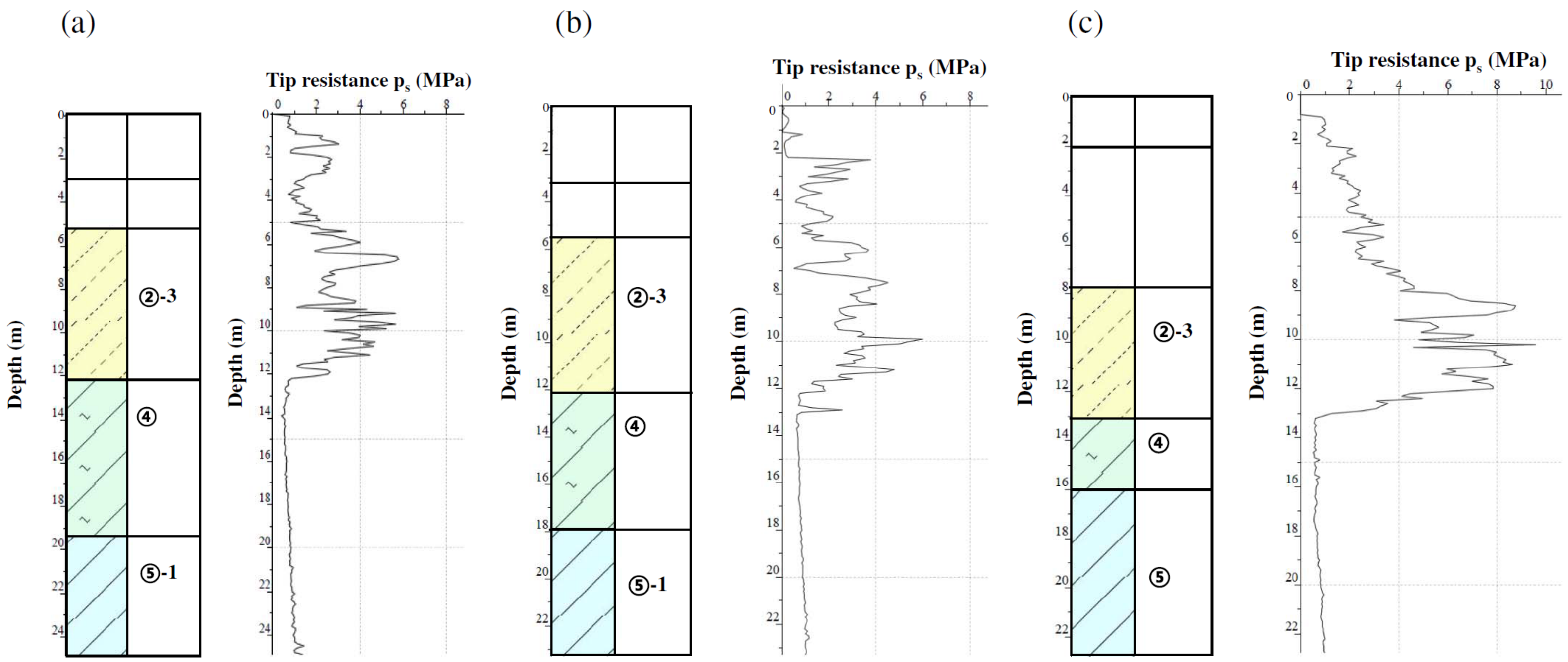
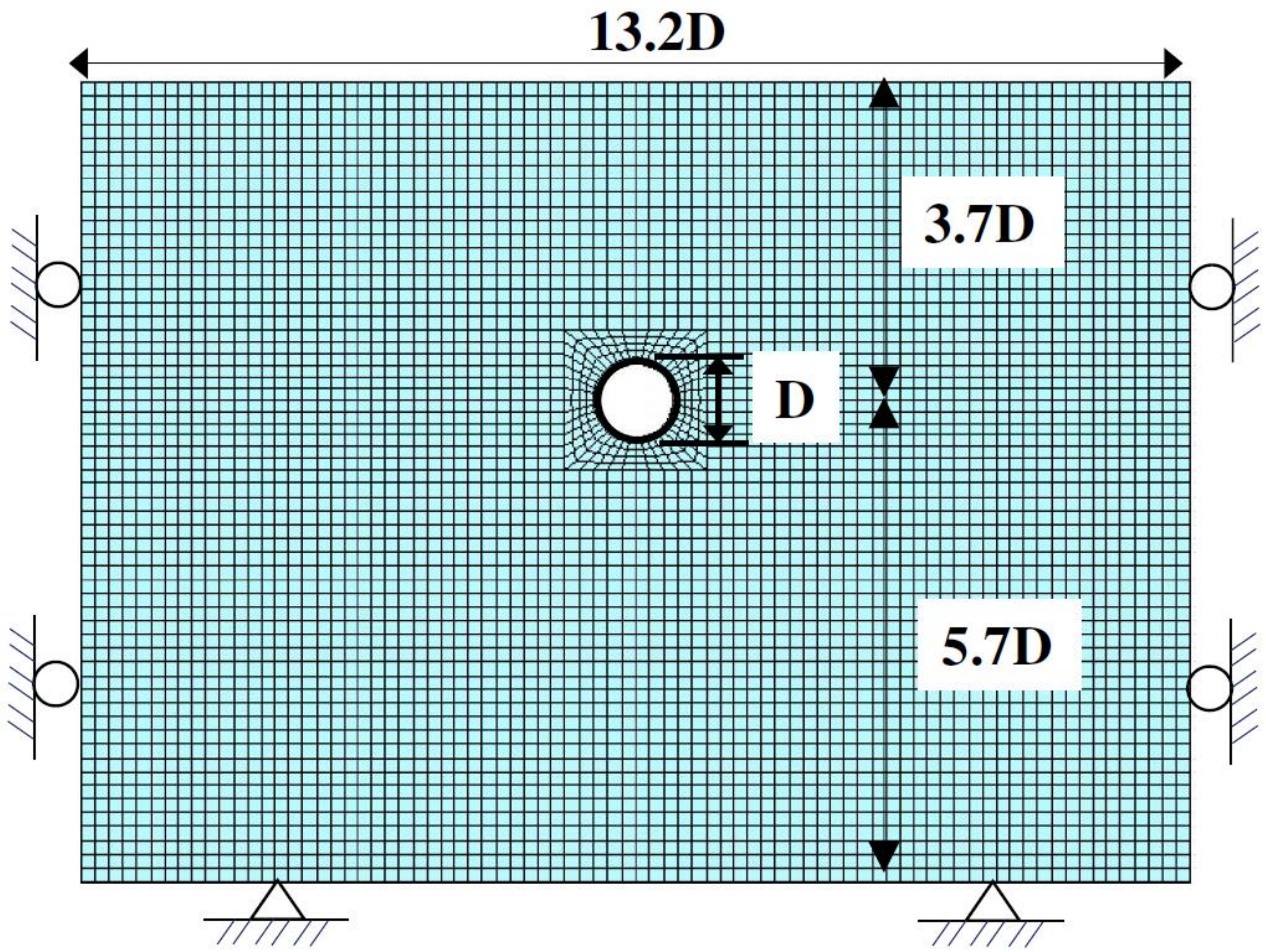
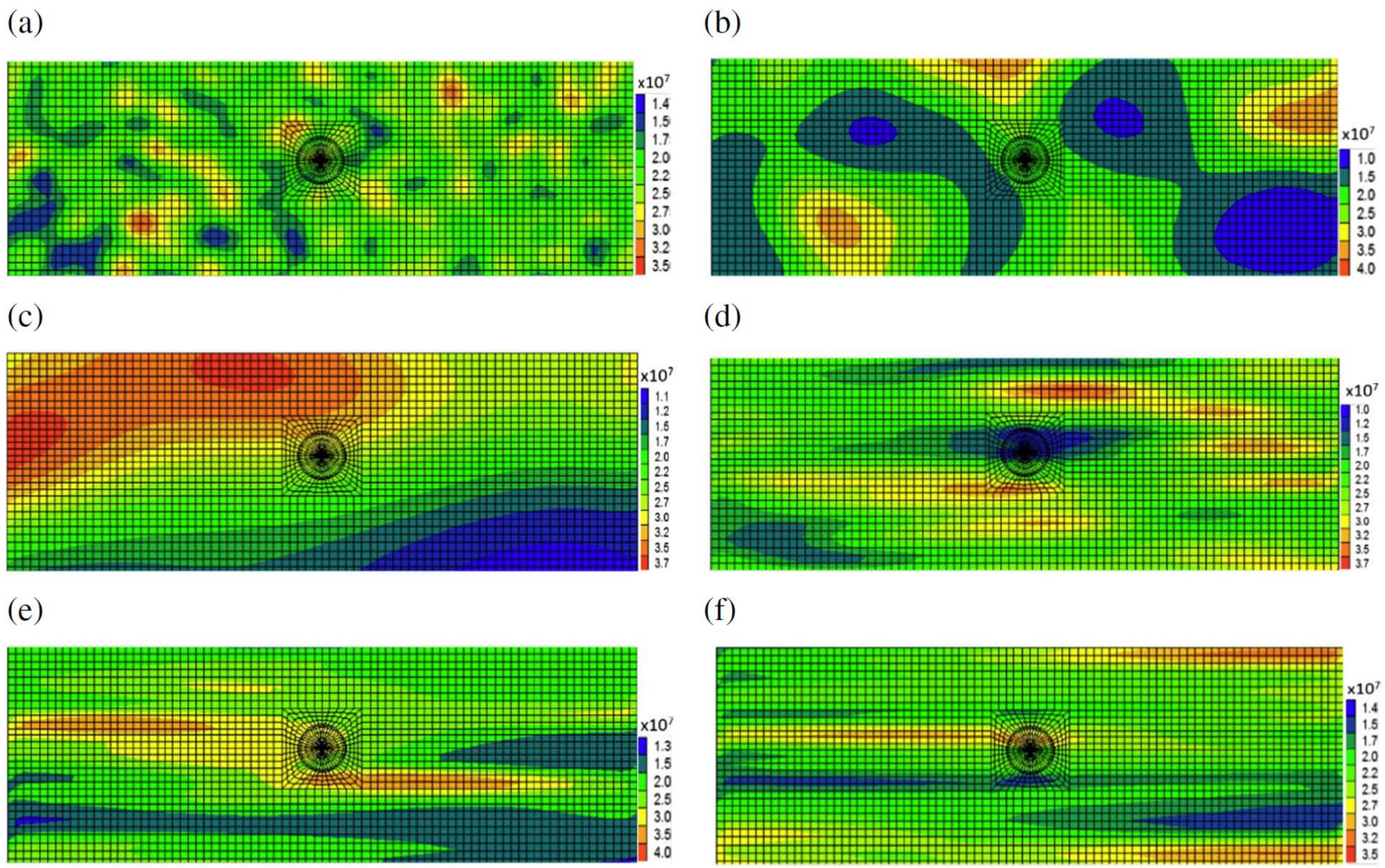


Fig. 3. Tip resistance from representative CPT tests obtained at Shanghai.

The Model

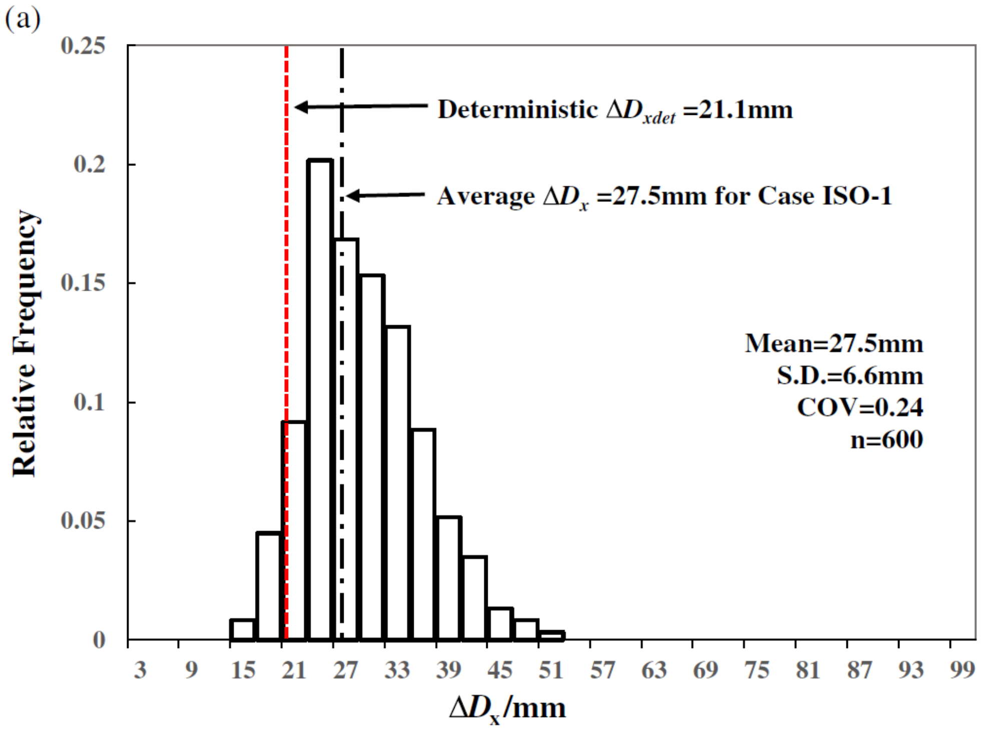


The Random field models

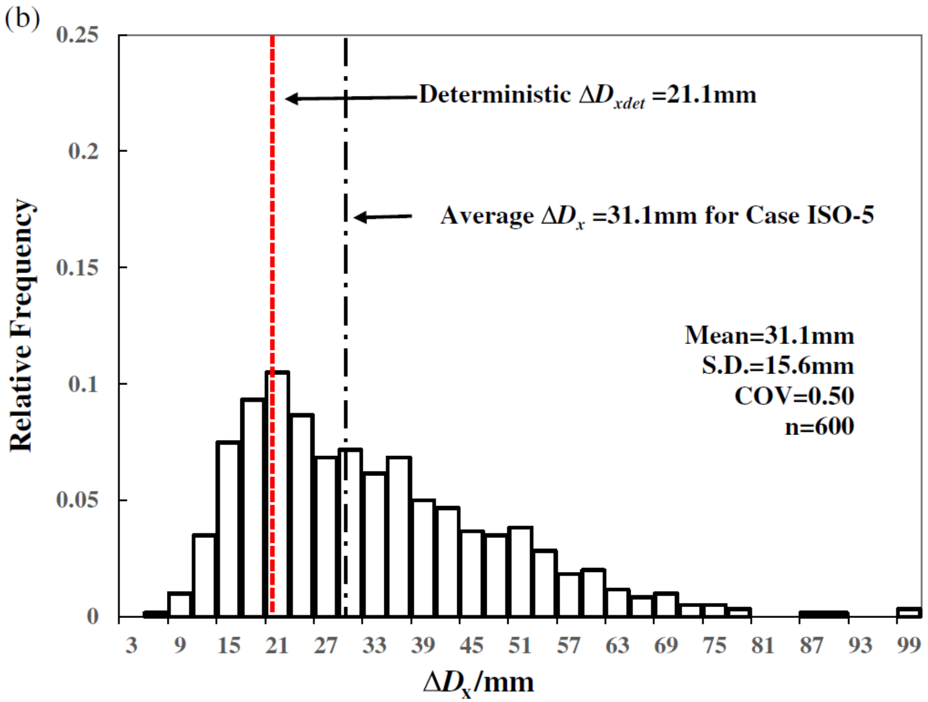


Results

It is clear that ignoring the spatial variability of soil's Young's modulus can leads to an underestimate of the tunnel convergence.



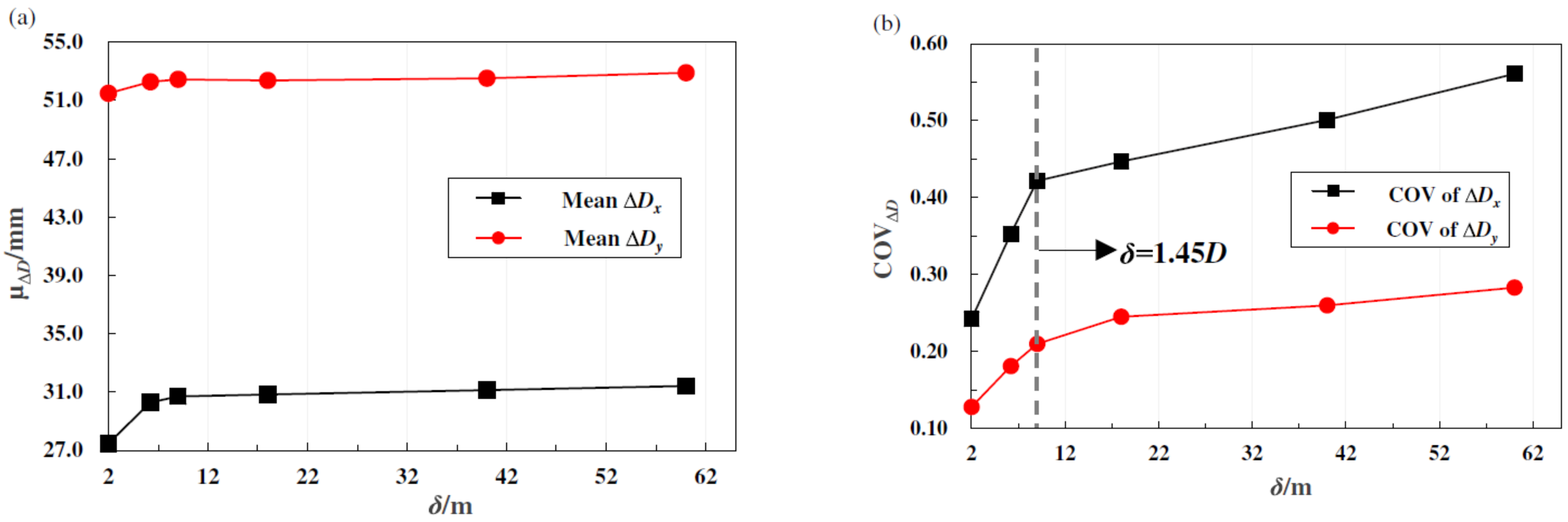
δ = 2 m



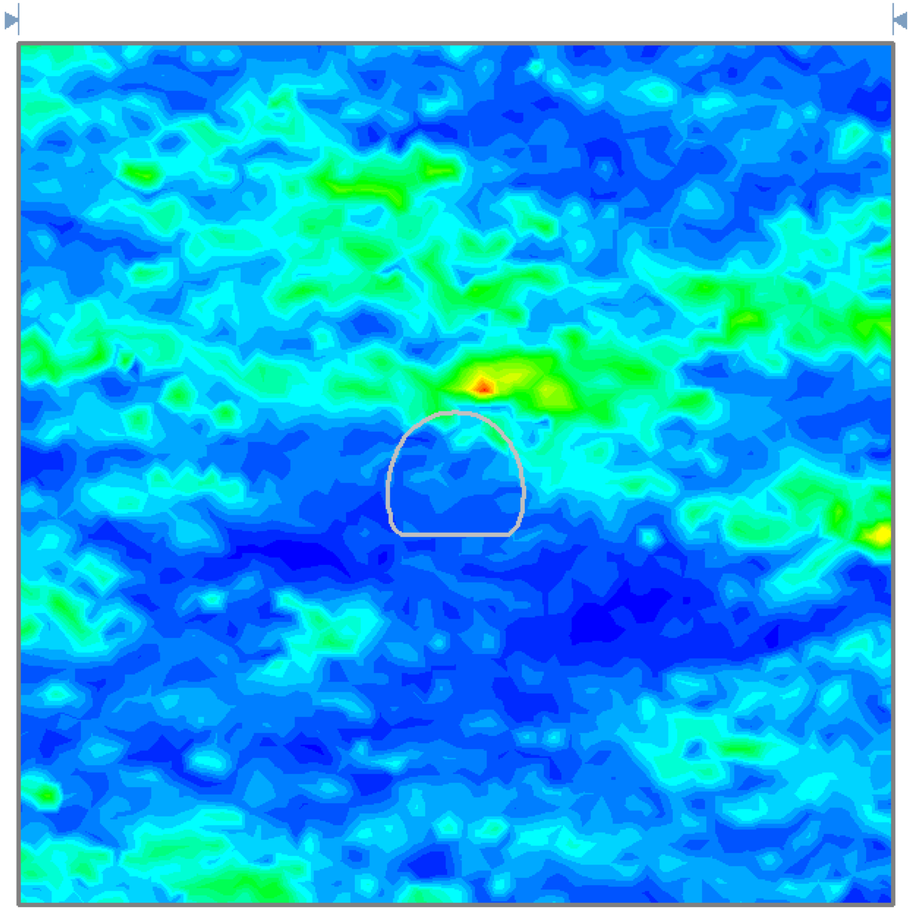
δ = 40 m

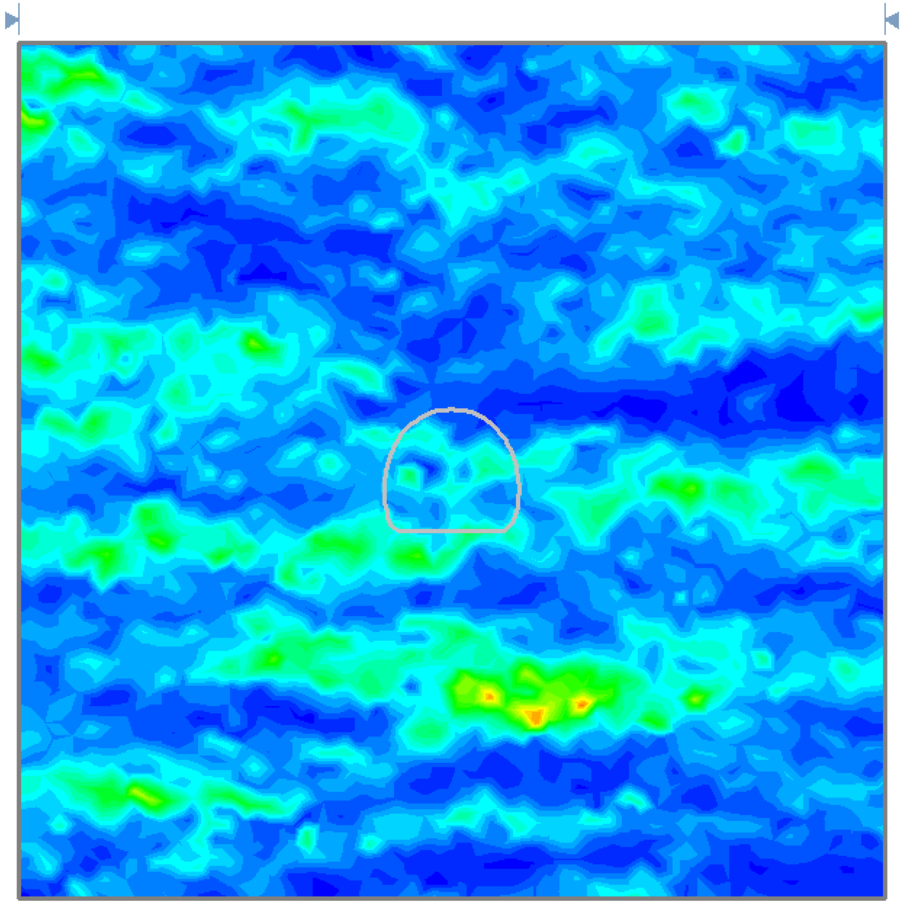
Results

It is clear that ignoring the spatial variability of soil’s Young's modulus can leads to an underestimate of the tunnel convergence.



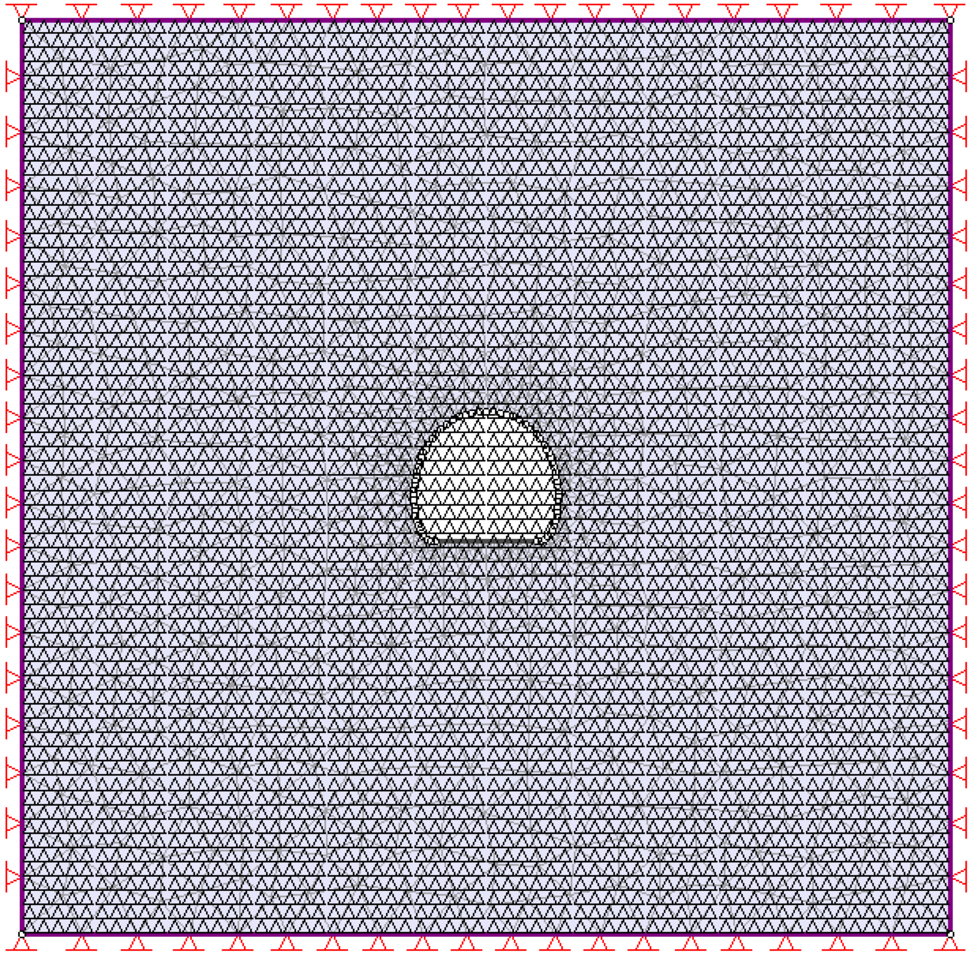
Example 2: Tunnel random field 1 & 7



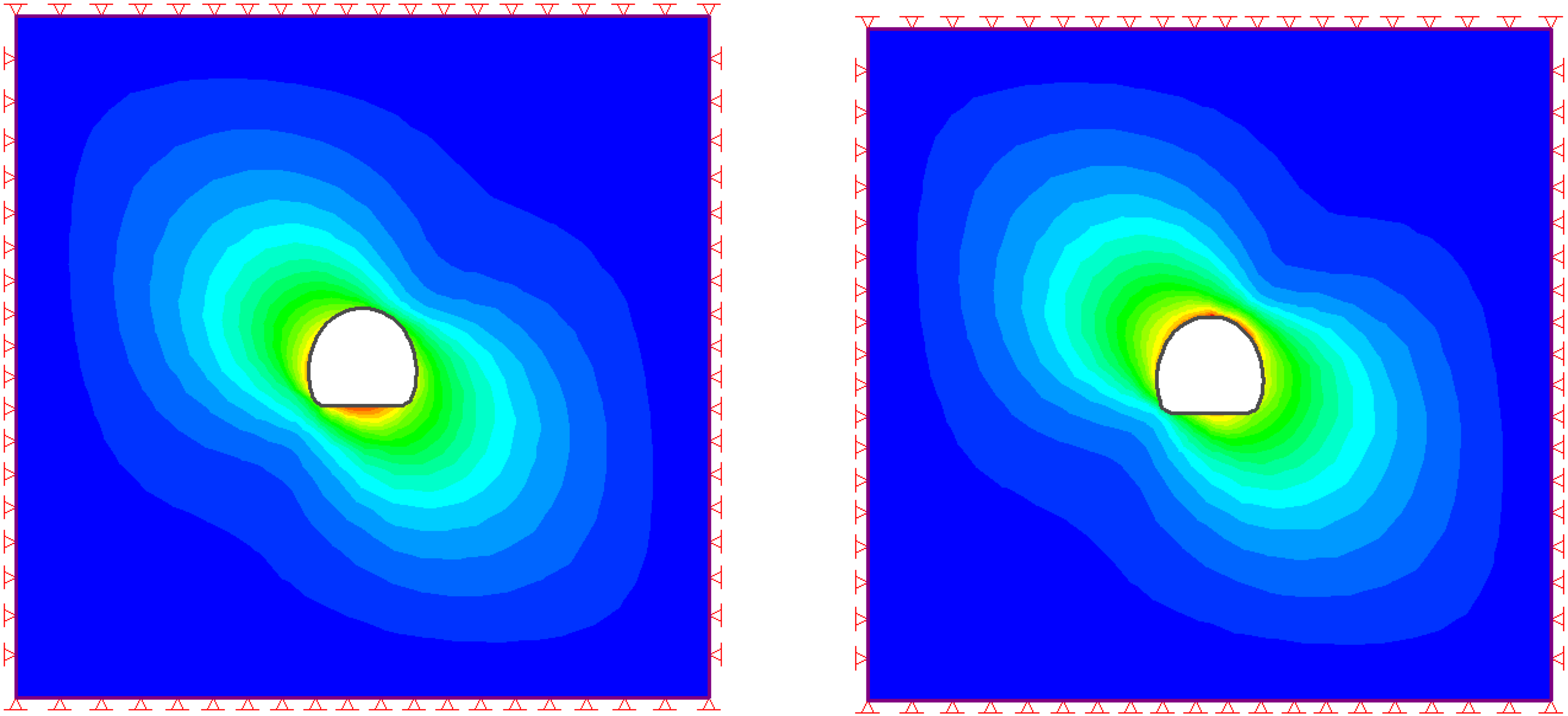


Example 2: Tunnel in Slide2 and RS2

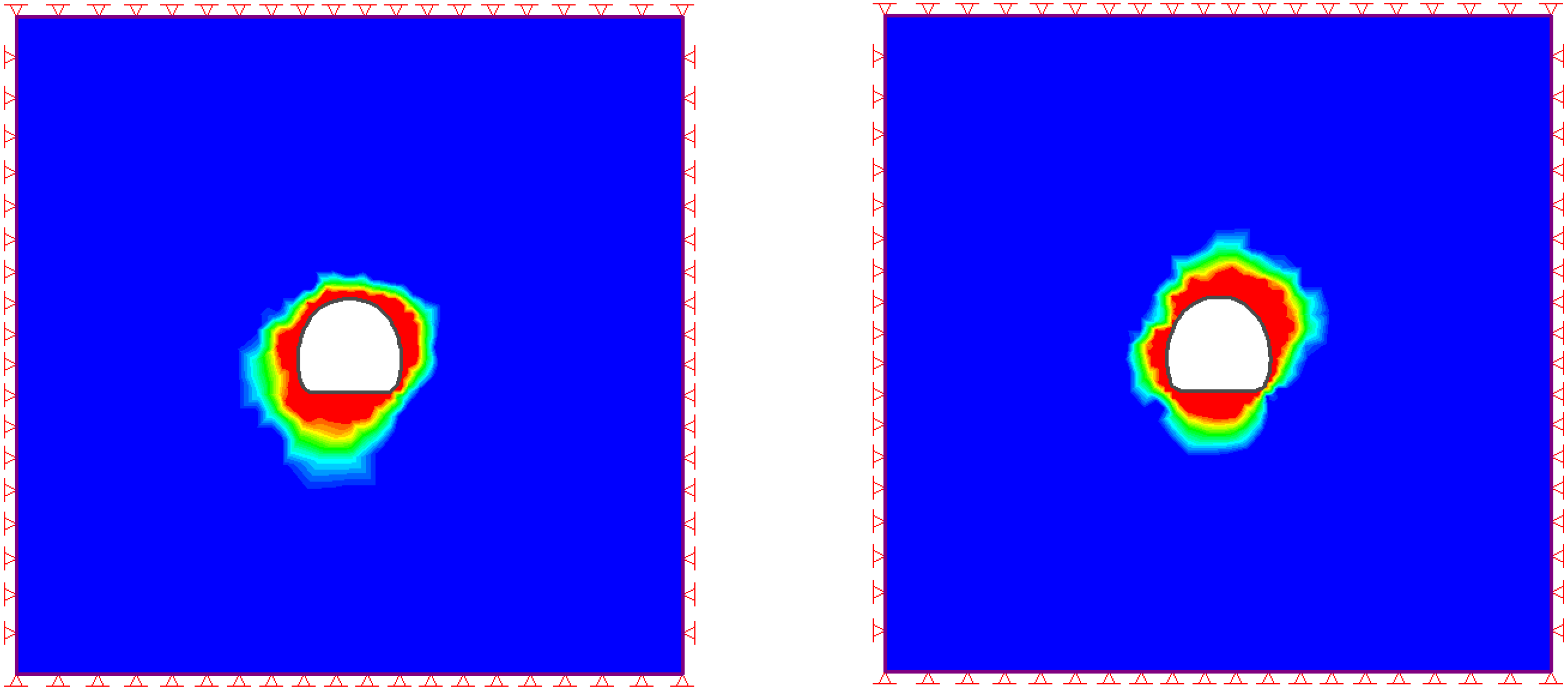
Spatial field, discrete function, RS2:



Total Displacement: Field 1 & 7



Yielded element: Field 1 & 7



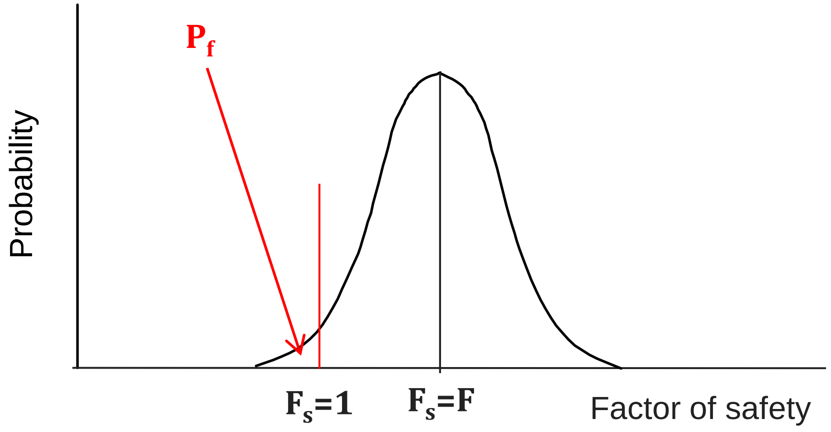
Reliability analysis of retaining walls

Section 7

Probabilistic analysis

Probabilistic analysis:

$\mathbf{X} = [X_1, X_2, X_3, \dots, X_n]$: Set of random variables



Performance function:

$$g(\mathbf{X}) = F_s - 1.0, \quad g(\mathbf{X}) = 0$$

F_s : Deterministic factor of safety

Probability of failure:

$$P_f = P[g(\mathbf{X}) \leq 0] = \int_{g(\mathbf{X}) \leq 0} f_{\mathbf{X}}(\mathbf{X}) d\mathbf{X}$$

Griffiths et al. (2010): Divided probabilistic slope stability analysis into four major methods

$$\left.\begin{array}{l}\text{1) Point Estimate Method (PEM)}\\ \text{2) First Order Second Moment Method (FOSM)}\\ \text{3) First Order Reliability Method (FORM)}\end{array}\right\} \text{Approximate method}$$

4) Monte Carlo simulation (MC)

Probabilistic methods of analysis

- Point Estimate Method (PEM):
- Probability distribution for input random variables can be replaced by discrete probability distribution having only two values with two associated probability concentration
- The mean value of Fs: $$\mu_{FS} = \sum_{i=1}^{2^n} P_i \left(FS_i \right)$$
- Standard deviation of Fs : $$\sigma_{FS}{}^2 = \sum_{i=1}^{2^n} P_i \left(FS_i \right)^2 - \mu_{FS}{}^2$$
- Pi are weighting coefficients: $$\left(\sum_{i=1}^{2^n} P_i = 1 \right)$$
- Reliability index can be calculated as: $$\beta = \frac{\mu_{FS} - 1}{\sigma_{FS}}$$
- Disadvantage: may lead to incorrect interpretation if performance function is highly nonlinear or the random variables are asymmetric

Probabilistic methods of analysis

First Order Second Moment (FOSM)

The mean and variance of the factor of safety are approximated by a first-order Taylor series expansion about the mean values of random variables

Reliability index can be calculated as:

$$\beta = \frac{FS\left(\mu_{x_i}\right) - 1}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial FS}{\partial X_i}\right)\left(\frac{\partial FS}{\partial X_j}\right)\mathrm{Cov}\left[X_i, X_j\right]}}$$

Disadvantage: reliability index depends on how the performance function is formulated

First Order Reliability Method (FORM)

Hasofer and Lind proposed mapping the random variable X from actual space to a normalized space

Reliability index is the minimum distance between the mean values and the limit state surface (g(x)=0)

It is assumed that the mean values of the random variables lie on the safe side of the performance function

$$\beta = \min_{g=0} \sqrt{\left\{\frac{X_i - \mu_i^N}{\sigma_i^N}\right\}^T [R]^{-1} \left\{\frac{X_i - \mu_i^N}{\sigma_i^N}\right\}} \qquad i=1,2,\ldots,n$$

Reliability of Simple linear limit states

Simple linear limit state function with one resistance term (R_m) and one load term (Q_m):

$$g = R_m - Q_m \qquad \text{Or} \qquad g = \boxed{\frac{R_m}{Q_m}} - 1 \;\longrightarrow\; F_s = \frac{R_m}{Q_m}$$

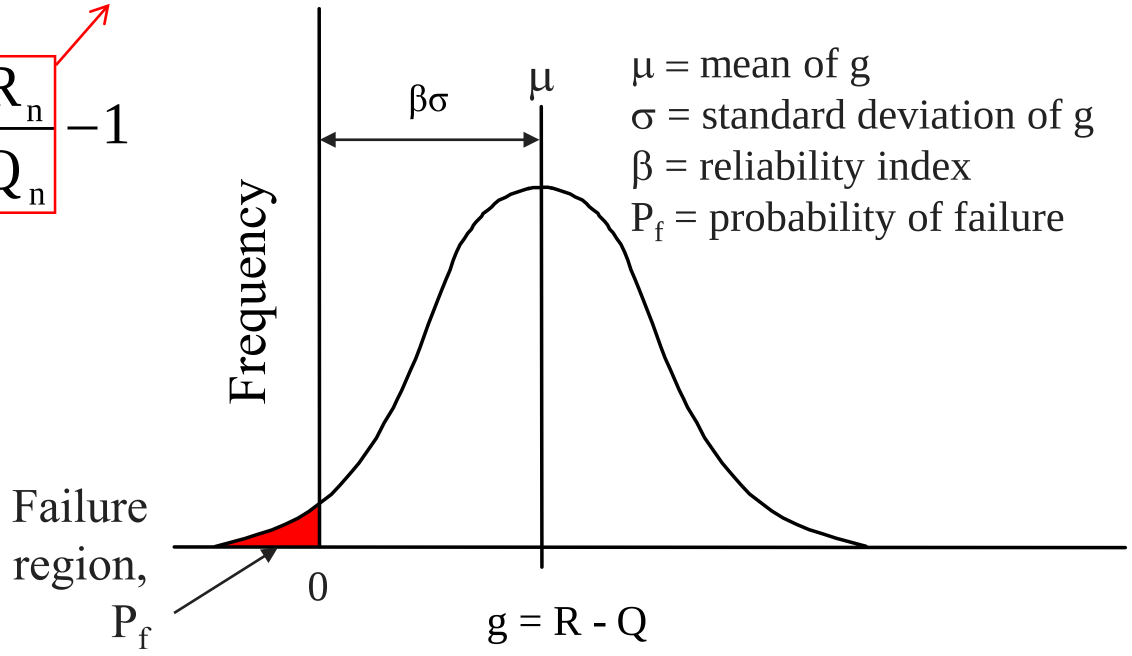
Most often, predicted (nominal) values of resistance (R_n) and load (Q_n) used in limit state design functions vary from measured values for soil-structure problems.

$$\lambda_R = \frac{R_m}{R_n} \qquad \lambda_Q = \frac{Q_m}{Q_n}$$

$$\Rightarrow \quad g = \underbrace{\boxed{\frac{\lambda_R R_n}{\lambda_Q Q_n}}}_{\text{OFS}} - 1$$

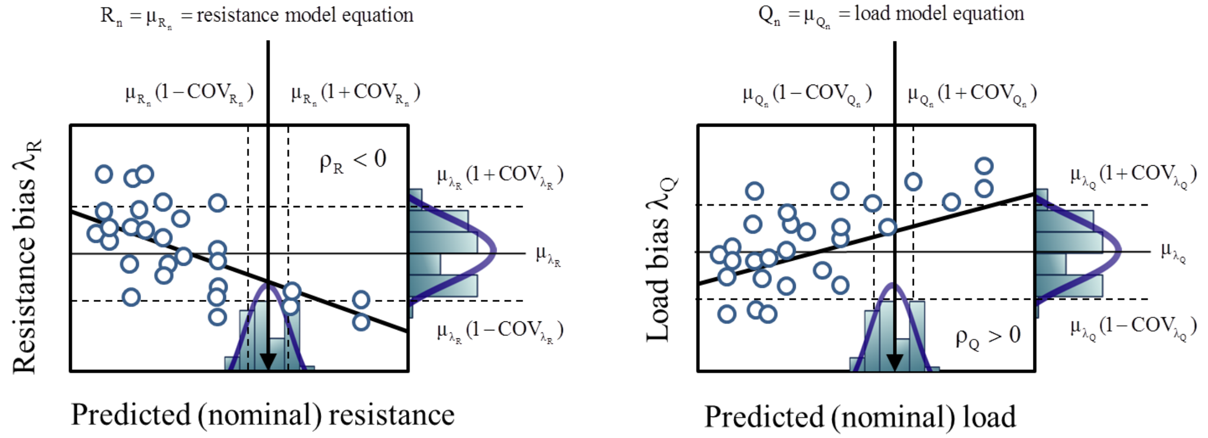
$$\Rightarrow \quad P(g < 0) = 1 - \Phi(\beta)$$

$$\Rightarrow \quad \beta = \left(\mu_{\ln\frac{R_m}{Q_m}} \Big/ \sigma_{\ln\frac{R_m}{Q_m}} \right)$$



Reliability of Simple linear limit states

$$g = \frac{\lambda_R}{\lambda_Q}\frac{R_n}{Q_n} - 1 \quad \textbf{Equation 6.6}$$



$$P\left(\frac{\lambda_R}{\lambda_Q}\frac{R_n}{Q_n} < 1\right) = \Phi(-\beta)$$

$$\beta = f(\mu_{R_n}, COV_{R_n}, \mu_{Q_n}, COV_{Q_n}, \mu_{\lambda_R}, COV_{\lambda_R}, \mu_{\lambda_Q}, COV_{\lambda_Q}, \rho_R, \rho_Q, \rho_{RQ})$$

Equation 6.12

$$\beta = \frac{\ln\left[\dfrac{\mu_{\lambda_R}\mu_{R_n}}{\mu_{\lambda_Q}\mu_{Q_n}}\sqrt{\dfrac{(1+COV_{Q_n}^2)(1+COV_{\lambda_Q}^2)}{(1+COV_{R_n}^2)(1+COV_{\lambda_R}^2)}}\right]}{\sqrt{\ln\left[(1+COV_{R_n}^2)(1+COV_{\lambda_R}^2)(1+COV_{Q_n}^2)(1+COV_{\lambda_Q}^2)(1+\rho_1 COV_{R_n}COV_{\lambda_R})^2(1+\rho_2 COV_{Q_n}COV_{\lambda_Q})^2/(1+\rho_3 COV_{R_n}COV_{Q_n})^2\right]}}$$

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Table S1. Example of levels of understanding for pullout limit state for steel reinforcement in MSE walls and adjustments to nominal load (Q_n) to consider construction and post-construction monitoring programs.

Level of understanding	Experience of designer with MSE walls of same type, component materials and complexity	Materials and component testing	Project complexity
High	Large number of MSE walls of the same type using same limits states design approach.	Project-specific pullout tests conducted and project-specific soil strength and unit weight verification testing carried out at time of construction. or Pullout test data available for similar soils and reinforcement materials that meet project specifications. Variability in project materials meeting specifications is small. Material specifications are tight and reflect high quality materials.	Wall geometry, boundary conditions, surcharge loading and reinforcement layout matching conditions for which load and pullout models were calibrated.
Typical	MSE walls of this type are designed routinely using same limits states design approach.	Pullout test data available for similar soils and reinforcement materials that meet project specifications. Variability in project materials meeting specifications is typical. Material specifications are tight and reflect typical guideline recommendations (e.g., **AASHTO 2014**).	Wall geometry, boundary conditions, surcharge loading and reinforcement layout matching conditions for which load and pullout models were calibrated.
Low	Limited or no experience with MSE wall type selected for project.	Pullout test data available for similar soils and reinforcement materials that meet project specifications. Variability in project materials meeting specifications is large. Material specifications are at the lower limit of typical guideline recommendations (e.g., **AASHTO 2014**).	Complex wall geometry and reinforcement layout not matching conditions for which load and pullout models were calibrated, but reasonable assumptions can be made to allow limit state models to be used.

Notes: 1) If quality of construction of project after design is expected to be lower than typical, then the magnitude of nominal load Q_n should be decreased in **Equation 1**. This is equivalent to increasing the design factor of safety in conventional ASD, or decreasing the target probability of failure (or increasing the target reliability index) in reliability-based design. 2) If a performance monitoring program is in place at time of construction and/or a post-construction monitoring program will be in place after construction, then the magnitude of nominal load Q_n can be increased in **Equation 1**. This is equivalent to decreasing the design factor of safety in conventional ASD, or increasing the target probability of failure (or decreasing the target reliability index) in reliability-based design.

Reliability of Simple linear limit states

The simple linear limit state design equation in the LRFD has the form:

$$\varphi R_n - \gamma_Q Q_n \geq 0 \quad \Rightarrow \quad \frac{\mu_{R_n}}{\mu_{Q_n}} = \frac{\gamma_Q}{\varphi}$$

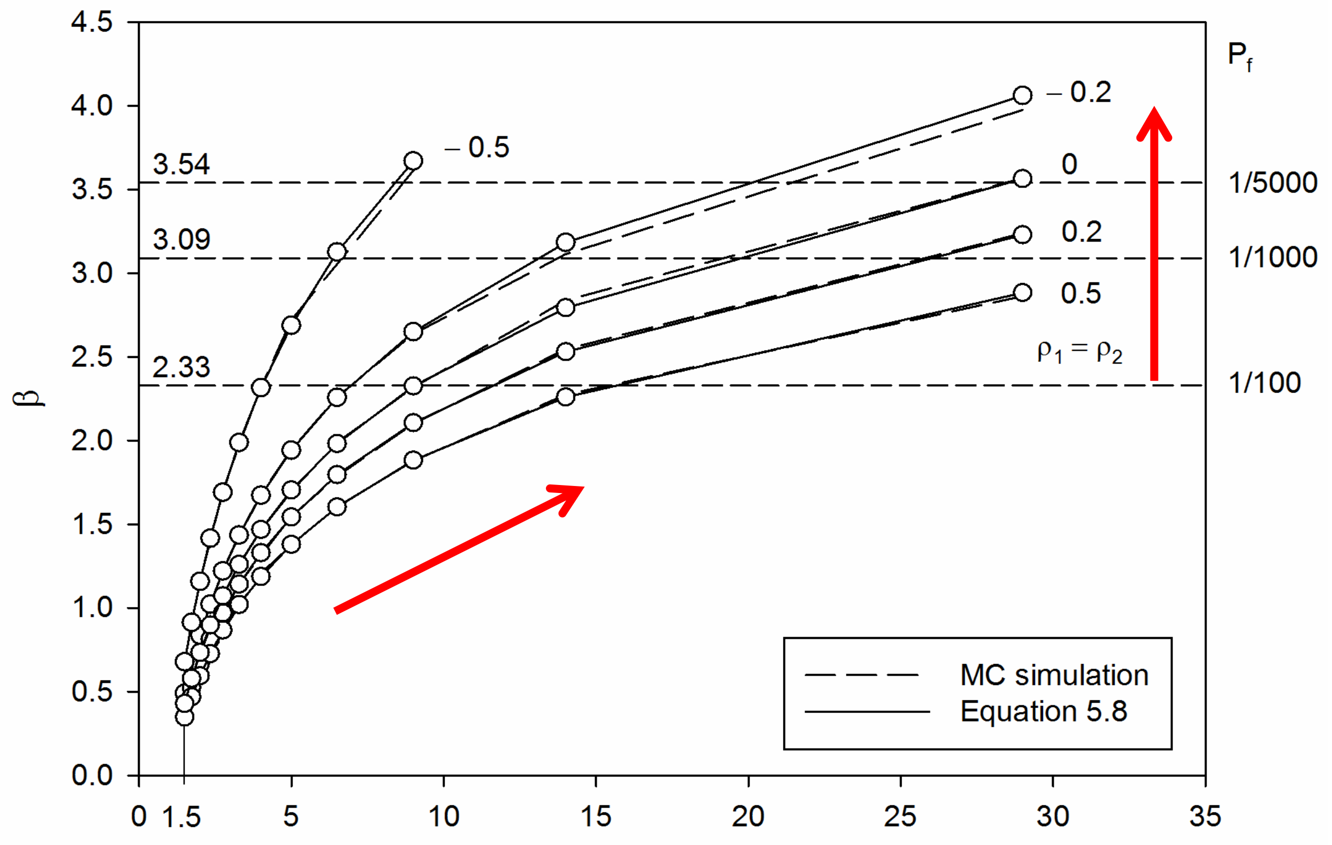
$$\beta = \frac{\ln\left[\dfrac{\mu_{\lambda_R}\gamma_Q}{\mu_{\lambda_Q}\varphi}\sqrt{\dfrac{(1+COV_{Q_n}^2)(1+COV_{\lambda_Q}^2)}{(1+COV_{R_n}^2)(1+COV_{\lambda_R}^2)}}\right]}{\sqrt{\ln\left[(1+COV_{R_n}^2)(1+COV_{\lambda_R}^2)(1+COV_{Q_n}^2)(1+COV_{\lambda_Q}^2)(1+\rho_R COV_{R_n}COV_{\lambda_R})^2(1+\rho_Q COV_{Q_n}COV_{\lambda_Q})^2\right]}}$$

$$\varphi = \gamma_Q \frac{\left(\dfrac{\mu_{\lambda_R}}{\mu_{\lambda_Q}}\right)\sqrt{\dfrac{(1+COV_{Q_n}^2)(1+COV_{\lambda_Q}^2)}{(1+COV_{R_n}^2)(1+COV_{\lambda_R}^2)}}}{\exp\left\{\beta\sqrt{\ln\left[(1+COV_{R_n}^2)(1+COV_{\lambda_R}^2)(1+COV_{Q_n}^2)(1+COV_{\lambda_Q}^2)(1+\rho_R COV_{R_n}COV_{\lambda_R})^2(1+\rho_Q COV_{Q_n}COV_{\lambda_Q})^2\right]}\right\}}$$

Bathurst, Javankhoshdel and Allen (2016)

Reliability of Simple linear limit states

Sensitivity analysis using the closed-form solution and MC simulation



$$OFS = \frac{\mu_{\lambda_R}}{\mu_{\lambda_Q}} F = \frac{\mu_{\lambda_R}\mu_{R_n}}{\mu_{\lambda_Q}\mu_{Q_n}}$$

Bathurst, Javankhoshdel (2016)

LRFD calibration for simple limit states

Table 6.1 Bias statistics for example cases

Case	Resistance (method) bias			Load (method) bias		
	$\mu_{\lambda R}$	$COV_{\lambda R}$	ρ_R	$\mu_{\lambda Q}$	$COV_{\lambda Q}$	ρ_Q
1	1.00	0.30	0.00	1.00	0.30	0.00
2	2.00	0.30	0.00	0.30	0.30	0.00
3	1.00	0.50	0.00	1.00	0.50	0.00
4	2.00	0.50	0.00	0.30	0.50	0.00
5	1.00	0.30	-0.70	1.00	0.30	-0.70
6	2.00	0.30	-0.70	0.30	0.30	-0.70
7	1.00	0.50	-0.70	1.00	0.50	-0.70
8	2.00	0.50	-0.70	0.30	0.50	-0.70
9	1.00	0.50	0.70	1.00	0.50	0.70
10	2.00	0.50	0.70	0.30	0.50	0.70

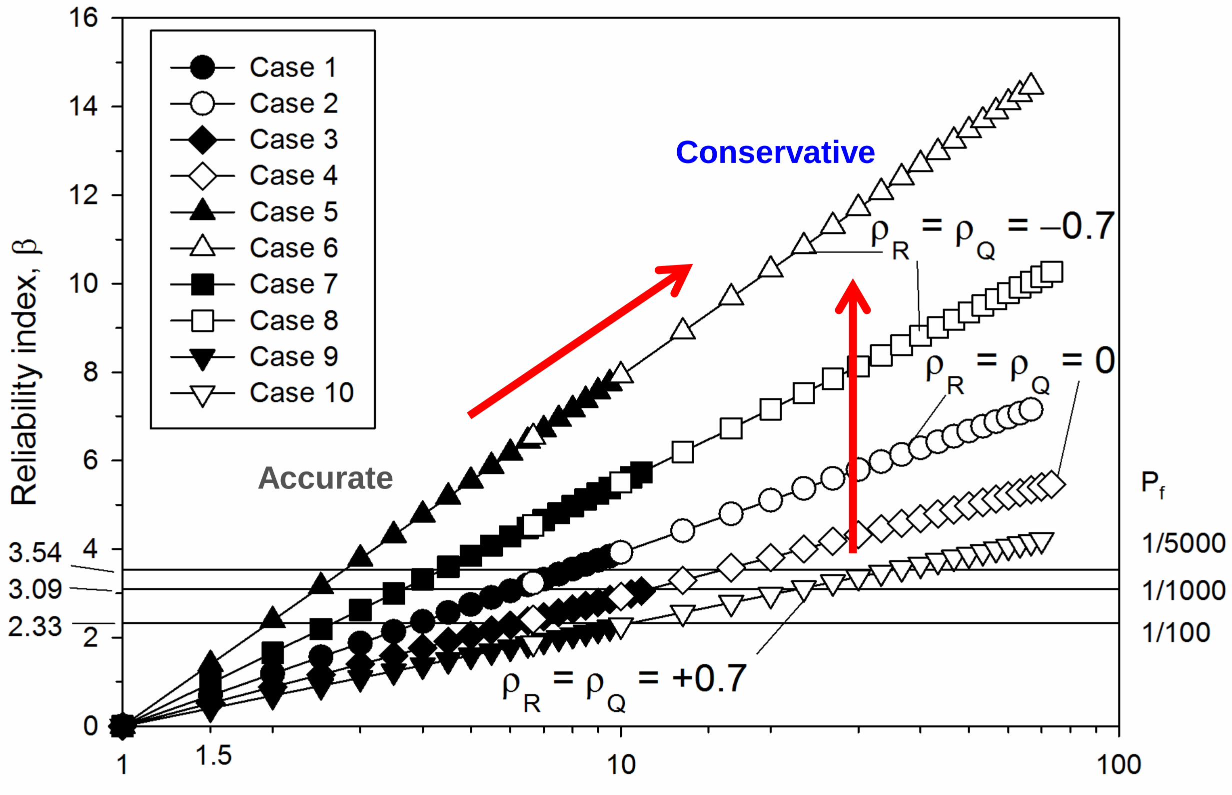
Accurate → 1

Conservative → 2

Bathurst et al. (2016)

LRFD calibration for simple limit states

Sensitivity analysis using the closed-form solution



Bathurst et al. (2016)

$$\mathrm{OFS} = \left(\frac{\mu_{\lambda_R} \gamma_Q}{\mu_{\lambda_Q} \varphi} \right)$$

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